APPENDIX A
FOURIER SERIES AND FOURIER INTEGRALS

A.1 FOURIER SERIES

This section discusses the harmonic analysis of signals and defines the Discrete Fourier Transform (DFT). For reasons which are given below, neither Fourier series nor the DFT are used when analyzing sampled EO imagers. However, the discussion in this section provides background for the Fourier integral theory described in the next section.

Harmonic analysis consists of projecting the observed signal *f(t)* onto a basis set of orthogonal sine and cosine functions. That is, a series of N sine and cosine functions is used to represent the observed signal. Assume that t is time, and the signal is observed for T seconds. Then the sines and cosines with periods equal to an integer sub-multiple of T seconds form the orthogonal basis set.

$$f(t) \approx \sum_{k=0}^{N} \left[a_k \cos\left(\frac{2\pi}{T} kt \right) + b_k \sin\left(\frac{2\pi}{T} kt \right) \right] . \tag{A.1}$$

The series in Equation A.1 is periodic with period T. In most practical cases, *f(t)* is not periodic. Equation A.1 gives the Fourier series for the infinitely repeated *f(t)*. See Figure A.1 for an illustration of this process. The finite, non-periodic function *f(t)* is made into a periodic function by replicating copies of itself.

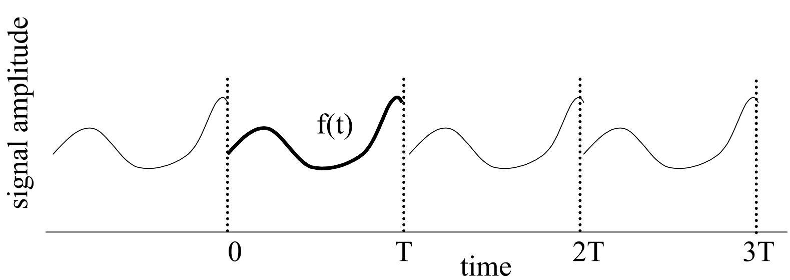


Figure A.1 The function $f(t)$ is replicated to form an infinitely extended, periodic function. The periodic extension of $f(t)$ often has discontinuities at the boundary between replicas.

The a_k and b_k coefficients are found by evaluating the following integrals.

$$\begin{aligned} a_k &= \frac{2}{T}\int_0^T f(t)\cos(\frac{2\pi}{T}kt)\,dt,\ k=0,\ 1,\ 2,\ 3,.\ .\ . \\ b_k &= \frac{2}{T}\int_0^T f(t)\sin(\frac{2\pi}{T}kt)\,dt,\ k=\ 1,\ 2,\ 3,.\ .\ .. \end{aligned} \tag{A.2}$$

Equations A.1 and A.2 can be cast in an exponential form which more resembles the Fourier transforms found in this book. The series is:

$$f(t) \approx \sum_{k=-N/2}^{N/2} \alpha_k e^{j2\pi\, kt/T} \tag{A.3}$$

where

$$\alpha_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi\, kt/T}\,dt \quad for\ \ k=0,\ \pm 1,\ \pm 2,\ \pm 3,\ .\ .\ . \tag{A.4}$$

and the α_k are complex numbers.

The Fourier series is a representation or approximation to the function *f(t)* in the interval $0 \le t \ge T$. In the frequency domain, each k/T corresponds to a discrete frequency. The function *f(t)* is made periodic and then decomposed into a set of sinusoidal waves. Equations A.1 and A.3 give the Fourier series for the infinitely repeated function.

Trigonometric functions are unique in that uniformly spaced samples over an integer number of periods form an orthogonal basis set. In the above equations, *f(t)* can be sampled, and the integrals can be turned into sums. Assume that N samples are taken in time T. Then:

$$\begin{aligned} a_k &= \sum_{n=0}^{N-1} f(\frac{nT}{N})\cos(2\pi\ \ knT/N)\ \ for\ \ k=0,\ 1,\ 2,\ ..,N/2 \\ b_k &= \sum_{n=0}^{N-1} f(\frac{nT}{N})\sin(2\pi\ \ knT/N)\ \ for\ \ k=0,\ 1,\ 2,\ ..,N/2. \end{aligned} \tag{A.5}$$

The representation of *f(t)* is now:

$$f(t) \approx \sum_{k=0}^{N/2}\left[a_k \cos\left(\frac{2\pi}{T}kt\right) + b_k \sin\left(\frac{2\pi}{T}kt\right)\right]. \tag{A.6}$$

It should be noted that the Equation A.6 approximation for *f(t)* is not the same as the Equation A.1 approximation. In Equation A.6, the a_k and b_k are found from the sampled values of *f(t)*.

Equations A.2, A.4, and A.5 represent discrete line spectra, because the a_k, b_k, and α_k are amplitudes of sinusoidal waves. However, Equations A.1, A.3, and A.6 are continuous representations of *f(t)*. It is often convenient to have discrete versions of both the frequency domain and space or time domain functions. One might assume that the samples of *f(t)* [the *f*(*nT*/*N*) samples in Equation A.5] can represent the function in time (space). However, in that case, the time (space) representations do not have an inverse Fourier relationship to the discrete frequency components. Remember that the series representation of *f(t)* is only an approximation.

However, the discrete time samples which result from sampling Equation A.6 have the correct relationship to the discrete frequency spectra in Equation A.5. Equations A.5 and A.7 form a discrete transform pair. That is, substituting *f'(nT/N)* from Equation A.7 into Equation A.5 yields the a_k and b_k which, when substituted into Equation A.7, yields the *f'(nT/N)* values.

$$f'(\frac{nT}{N}) = \sum_{k=0}^{N/2}\left[a_k \cos\left(\frac{2\pi}{N}kn\right) + b_k \sin\left(\frac{2\pi}{N}kn\right)\right] \quad for \;\; n = 0,\ 1,\ .\ .\ .\ ,\ N-1\ . \tag{A.7}$$

The fidelity with which the Fourier series in Equations A.1, A.3, A.6, or A.7 represents *f(t)* depends on many factors. Certainly *k* must be large enough to span the frequency content in *f(t)*. Too few terms in the series leads to errors in the approximation. Also, the Gibb's phenomenon will lead to considerable error in the region of any discontinuity in *f(t)*. Remember that *f(t)* is made into a periodic function by replicating copies of itself. As illustrated in Figure A.1, there is normally a discontinuity at the border between replicas of *f(t)*. Other errors arise because the Fourier series only represents a finite number of frequencies. Frequencies in *f(t)* not represented in the series will either "leak" and appear as a wrong frequency component or will not be included in the series representation.

Also, although the discrete transform defined by Equations A.5 and A.7 is widely used with sampled data, neither the DFT nor the Fourier series provide insight into the sampling characteristics of the system. The DFT depends on the sample values and does not express the Fourier transform of a sampled version of *f(t)* in terms of the pre-sample MTF, the sample rate, and the post-sample MTF.

The Fourier integral transform provides a more accurate representation of a non-periodic signal than does the Fourier series. Also, the integral transform provides the flexibility to express the Fourier transform of a sampled *f(t)* in terms of the pre- and post-sample MTFs and sample rate.

A.2 FOURIER INTEGRAL

Equations A.3 and A.4 are generalized to allow any frequency and to permit *f(t)* to be non-periodic. The sums become integrals, and the $-T/2 \le t \ge T/2$ interval now extends over all time.

$$F(\xi) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi\xi t} dt \tag{A.8}$$

$$f(t) = \int_{-\infty}^{\infty} F(\xi)e^{j2\pi\xi t}df. \tag{A.9}$$

Equations A.8 and A.9 define the Fourier integral transform pair, where ξ is frequency in Hertz. If t represents angle in milliradians rather than time, then ξ is frequency in cycles per milliradian. If t represents a spatial coordinate in units of millimeters, then ξ is frequency in cycles per millimeter.

The condition for existence of $F(\xi)$ is generally given as:

$$\int_{-\infty}^{\infty} |f(t)|dt \ \langle \ \infty \tag{A.10}$$

or, in other words, *f(t)* must be absolutely integrable. However, Equation A.10 is sufficient but not necessary; *f(t)* often exists even if this condition is not met.

Three properties of the Fourier transform are used often in this book. The first property is linearity. If $F(\xi)$ is the Fourier transform of *f(t),* and $G(\xi)$ is the Fourier transform of *g(t)*, then the Fourier transform of [*f(t)* + *g(t)*] is [$F(\xi)$ + $G(\xi)$]. The second property is time-shifting. If $F(\xi)$ is the Fourier transform of *f(t)*, then the Fourier transform of $f(t-\tau)$ is $F(\xi)e^{-j2\pi\xi\tau}$.

The third property is that if *f(x,y)* is separable, then the Fourier transform of *f(x,y)* is separable. That is, if:

$$f(x,y) = g(x)h(y), \tag{A.11}$$

then

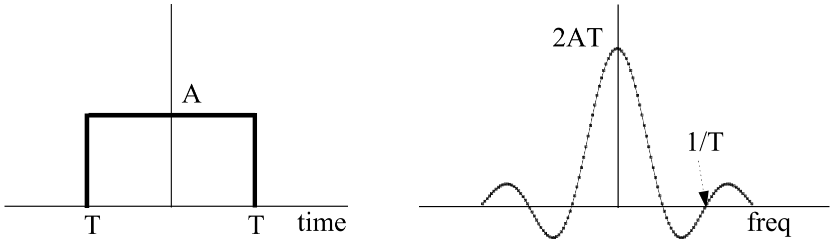
$$F(\xi,\eta) = G(\xi)H(\eta). \tag{A.12}$$

Another theorem used throughout the book is that a convolution in the space domain is a multiplication in the frequency domain. That is:

$$\begin{aligned} &\text{If} \quad h(t) = \int_{-\infty}^{\infty} f(\tau)g(\tau - t)d\tau \\ &\text{then} \quad H(\xi) = F(\xi)G(\xi). \end{aligned} \tag{A.13}$$

An asterisk is used in this book to represent convolution. For example, $h(t) = f(t) * g(t)$.

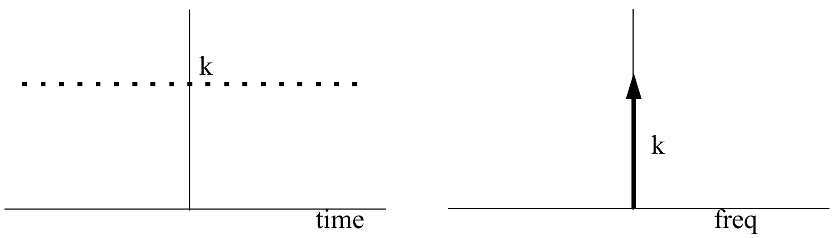
Some Fourier transform pairs are shown in the figures below. The symbol $\Leftrightarrow$ indicates the transform pair relationship. In Figure A.2, the transform of a rect function is a sinc function. In Figure A.3, the transform of a constant is a delta function. In Figure A.4, the transform of a Gaussian is a Gaussian.



$$f(t) = A \quad |t| \langle T$$
$$= \frac{A}{2} \quad |t| = T$$
$$= 0 \quad |t| \rangle T$$

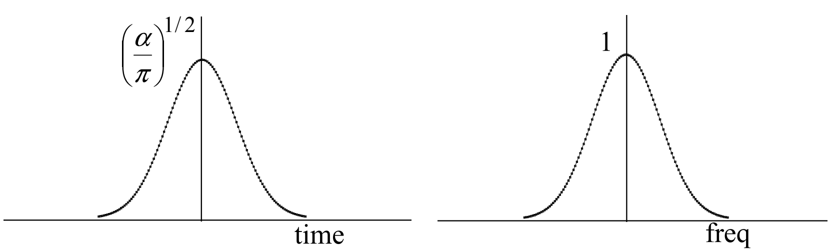
$$\Leftrightarrow \qquad F(\xi) = 2AT\frac{\sin(2\pi\xi T)}{2\pi\xi T}$$

Figure A.2 Rect and sinc wave transform pair.



$$f(t) = k \qquad \Leftrightarrow \qquad F(\xi) = k\delta(\xi)$$

Figure A.3 The transform of a constant is a delta function centered at the frequency origin.



$$f(t) = \left(\frac{\alpha}{\pi}\right)^{1/2} e^{-\alpha t^2} \qquad \Leftrightarrow \qquad F(\xi) = e^{\pi^2 \xi^2 / \alpha}$$

Figure A.4 The transform of a Gaussian is a Gaussian.

APPENDIX B
THE IMPULSE FUNCTION

B.1 DEFINITION

The impulse function is also known as the Dirac delta function. It can be described as a function that has an infinite height, zero width and an area equal to unity. Mathematically, the impulse function is defined by Equation B.1.

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0). \tag{B.1}$$

For practical purposes, we can define δ(x) as follows.

$$\delta(x) = \lim_{b \to 0} \frac{1}{|b|} Gauss(\frac{x}{b}). \tag{B.2}$$

The area under the Gaussian function must remain unity as *b* gets smaller, so the height of the function increases as shown in Figure B.1. The practical definition given in Equation B.2 could have used the rectangle or a number of other shapes. The important concept is that the impulse function has zero width but unity area.

B.2 PROPERTIES OF THE IMPULSE FUNCTION

There are a few important properties of the impulse response that are used frequently throughout this text. One of the defining properties of the impulse function is

$$\delta(x - x_o) = 0\ ,\ x \neq x_o \tag{B.3}$$

so that the only location where the impulse function has a non-zero value is at the location of the impulse function. Another defining property is the integral property of the impulse function

$$\int_{-\infty}^{\infty} \delta(x - x_o)\, dx = 1 \tag{B.4}$$

which simply states that the area of an impulse function is 1. The *sifting* property is described as such because the impulse function "sifts" out the value of a function at a particular point

$$\int_{x_1}^{x_2} f(x)\delta(x-x_o)dx = f(x_o) \qquad x_1 < x_o < x_2 . \tag{B.5}$$

The impulse function is an even function, so that

$$\delta(x) = \delta(-x) . \tag{B.6}$$

The *comb* function is an infinite set of equally spaced delta functions. The *comb* function has been described with many different notations including Bracewell's "shah" function. The Fourier transform of a comb function in space is a comb function in frequency. If X is the spacing between delta functions, then

$$\sum_{n=-\infty}^{\infty} \delta(x\text{-}nX) \Leftrightarrow \sum_{n=-\infty}^{\infty} \delta(\xi\text{-}n/X) \tag{B.7}$$

where $\Leftrightarrow$ indicates a Fourier transform pair.

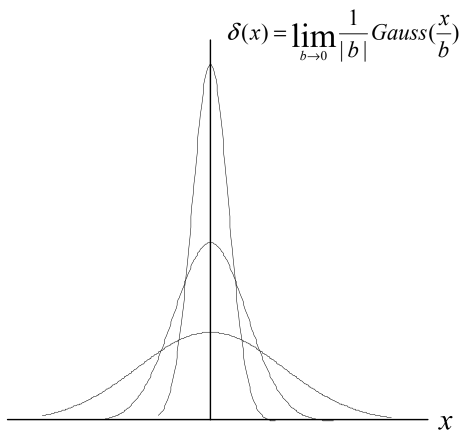
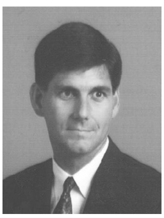


Figure B.1 As the width of the Gaussian decreases, the height increases in order to maintain the area under the curve at unity.

INDEX

Mr. Richard H. Vollmerhausen currently heads the Model Development Branch at the US Army's Night Vision Lab. The branch is updating the Army's target acquisition models to include the effect of sampling on performance and to make other model enhancements in order to predict the performance of advanced technology sensors. During his tenure at NVL, he was a project engineer and EO systems analyst for numerous Army weapon systems. His previous work included designing air-to-air missile seekers for the Navy and working as an Instrumentation Engineer for Douglas Aircraft on the Saturn/Apollo Program. Mr. Vollmerhausen is the author of two books on electro-optical systems analysis and has published numerous journal and symposium papers.



Dr. Ronald G. Driggers has 12 years of electro-optics experience and has worked for or consulted to Lockheed Martin, SAIC, EOIR Measurements, Amtec Corporation, Joint Precision Strike Demonstration Project Office, and Redstone Technical Test Center. He is currently working for the US Army's Night Vision and Electronic Sensors Directorate and is the U.S. representative to the NATO panel on advanced thermal imager characterization. Dr. Driggers is the author of two books on infrared and electro-optics systems and has published over 30 refereed journal papers. He is editor of Marcel Dekker's *Encyclopedia of Optical Engineering* and is an Associate Editor of *Optical Engineering*.