

Field Guide to

**Probability,
Random Processes, and
Random Data Analysis**

Larry C. Andrews
Ronald L. Phillips

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Welcome to the *SPIE Field Guides*—a series of publications written directly for the practicing engineer or scientist. Many textbooks and professional reference books cover optical principles and techniques in depth. The aim of the *SPIE Field Guides* is to distill this information, providing readers with a handy desk or briefcase reference that provides basic, essential information about optical principles, techniques, or phenomena, including definitions and descriptions, key equations, illustrations, application examples, design considerations, and additional resources. A significant effort will be made to provide a consistent notation and style between volumes in the series.

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Field Guide to Probability, Random Processes, and Random Data Analysis

Developed in basic courses in engineering and science, mathematical theory usually involves deterministic phenomena. Such is the case for solving a differential equation that describes a linear system where both input and output are deterministic quantities. In practice, however, the input to a linear system, such as imaging or radar systems, can contain a “random” quantity that yields uncertainty about the output. Such systems must be treated by probabilistic rather than deterministic methods. For this reason, probability theory and random-process theory have become indispensable tools in the mathematical analysis of these kinds of engineering systems.

Topics included in this *Field Guide* are basic probability theory, random processes, random fields, and random data analysis. The analysis of random data is less well known than the other topics, particularly some of the tests for stationarity, periodicity, and normality.

Much of the material is condensed from the authors’ earlier text *Mathematical Techniques for Engineers and Scientists* (SPIE Press, 2003). As is the case for other volumes in this series, it is assumed that the reader has some basic knowledge of the subject.

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Table of Contents

| | |
|--|----------|
| Glossary of Symbols and Notation | x |
| Probability: One Random Variable | |
| Terms and Axioms | 2 |
| Random Variables and Cumulative Distribution | 3 |
| Probability Density Function | 4 |
| Expected Value: Moments | 5 |
| Example: Expected Value | 6 |
| Expected Value: Characteristic Function | 7 |
| Gaussian or Normal Distribution | 8 |
| Other Examples of PDFs: Continuous RV | 9 |
| Other Examples of PDFs: Discrete RV | 12 |
| Chebyshev Inequality | 13 |
| Law of Large Numbers | 14 |
| Functions of One RV | 15 |
| Example: Square-Law Device | 16 |
| Example: Half-Wave Rectifier | 17 |
| Conditional Probabilities | |
| Conditional Probability: Independent Events | 19 |
| Conditional CDF and PDF | 20 |
| Conditional Expected Values | 21 |
| Example: Conditional Expected Value | 22 |
| Probability: Two Random Variables | |
| Joint and Marginal Cumulative Distributions | 24 |
| Joint and Marginal Density Functions | 25 |
| Conditional Distributions and Density Functions | 26 |
| Example: Conditional PDF | 27 |
| Principle of Maximum Likelihood | 28 |
| Independent RVs | 29 |
| Expected Value: Moments | 30 |
| Example: Expected Value | 31 |
| Bivariate Gaussian Distribution | 32 |
| Example: Rician Distribution | 33 |
| Functions of Two RVs | 34 |
| Sum of Two RVs | 35 |
| Product and Quotient of Two RVs | 36 |
| Conditional Expectations and Mean-Square Estimation | 37 |

Table of Contents

| | |
|--|-----------|
| Sums of N Complex Random Variables | 38 |
| Central Limit Theorem | 39 |
| Example: Central Limit Theorem | 40 |
| Phases Uniformly Distributed on $(-\pi, \pi)$ | 41 |
| Phases Not Uniformly Distributed on $(-\pi, \pi)$ | 42 |
| Example: Phases Uniformly Distributed on $(-\alpha, \alpha)$ | 43 |
| Central Limit Theorem Does Not Apply | 45 |
| Example: Non-Gaussian Limit | 46 |
| | |
| Random Processes | 48 |
| Random Processes Terminology | 49 |
| First- and Second-Order Statistics | 50 |
| Stationary Random Processes | 51 |
| Autocorrelation and Autocovariance Functions | 52 |
| Wide-Sense Stationary Process | 53 |
| Example: Correlation and PDF | 54 |
| Time Averages and Ergodicity | 55 |
| Structure Functions | 56 |
| Cross-Correlation and Cross-Covariance Functions | 57 |
| Power Spectral Density | 58 |
| Example: PSD | 59 |
| PSD Estimation | 60 |
| Bivariate Gaussian Processes | 61 |
| Multivariate Gaussian Processes | 62 |
| Examples of Covariance Function and PSD | 63 |
| Interpretations of Statistical Averages | 64 |
| | |
| Random Fields | 65 |
| Random Fields Terminology | 66 |
| Mean and Spatial Covariance Functions | 67 |
| 1D and 3D Spatial Power Spectrums | 68 |
| 2D Spatial Power Spectrum | 69 |
| Structure Functions | 70 |
| Example: PSD | 71 |
| | |
| Transformations of Random Processes | 72 |
| Memoryless Nonlinear Transformations | 73 |
| Linear Systems | 74 |
| Expected Values of a Linear System | 75 |

Table of Contents

| | |
|--|-----------|
| Example: White Noise | 76 |
| Detection Devices | 77 |
| Zero-Crossing Problem | 78 |
| Random Data Analysis | 79 |
| Tests for Stationarity, Periodicity, and Normality | 80 |
| Nonstationary Data Analysis for Mean | 81 |
| Analysis for Single Time Record | 82 |
| Runs Test for Stationarity | 83 |
| Equation Summary | 85 |
| Bibliography | 90 |
| Index | 91 |

Glossary of Symbols and Notation

| | |
|--|---|
| a, \mathbf{x}, \mathbf{u}, etc. | Random variable, process, or field |
| $B_{\mathbf{u}}(R)$ | Autocovariance or covariance function of random field |
| $C_{\mathbf{x}}(\tau)$ | Autocovariance or covariance function of random process |
| $C_{\mathbf{xy}}(\tau)$ | Cross-covariance function |
| CDF | Cumulative distribution function |
| Cov | Covariance |
| $D_{\mathbf{x}}(\tau)$ | Structure function |
| $E[.]$ | Expectation operator |
| $E[g(\mathbf{x}) A]$ | Conditional expectation operator |
| $f_{\mathbf{x}}(x), f_{\mathbf{x}}(x, t)$ | Probability density function |
| $f_{\mathbf{x}}(x A)$ | Conditional probability density |
| $F_{\mathbf{x}}(x), F_{\mathbf{x}}(x, t)$ | Cumulative distribution function |
| $F_{\mathbf{x}}(x A)$ | Conditional cumulative distribution function |
| ${}_pF_q$ | Generalized hypergeometric function |
| $h(t)$ | Impulse response function |
| $H(\omega)$ | Transfer function |
| $I_p(x)$ | Modified Bessel function of the first kind |
| $J_p(x)$ | Bessel function of the first kind |
| $K_p(x)$ | Modified Bessel function of the second kind |
| $m, m(t)$ | Mean value |
| m_k | k 'th standard statistical moment |
| $n!$ | Factorial function |
| PDF | Probability density function |
| Pr | Probability |
| $Pr(B A)$ | Conditional probability |
| PSD | Power spectral density |
| RV | Random variable |
| $R_{\mathbf{x}}(\tau)$ | Autocorrelation or correlation function |
| $R_{\mathbf{xy}}(\tau)$ | Cross-correlation function |
| $\mathfrak{R}_{\mathbf{x}}(\tau)$ | Long-time-average correlation function |
| $S_{\mathbf{x}}(\omega), S_{\mathbf{u}}(\kappa)$ | Power spectral density function |
| $U(x - a)$ | Unit step function |

Glossary of Symbols and Notation

| | |
|-----------------------------------|---|
| Var | Variance |
| $\text{Var}[\mathbf{x} A]$ | Conditional variance |
| $\overline{\mathbf{x}(t)}$ | Time average |
| z^* | Complex conjugate of z |
| $\gamma(c, x)$ | Incomplete gamma function |
| $\Gamma(x)$ | Gamma function |
| $\delta(x - a)$ | Dirac delta function (impulse function) |
| μ_k | k 'th central statistical moment |
| $\hat{\mu}(t)$ | Estimator of mean value |
| $\sigma^2, \sigma_{\mathbf{x}}^2$ | Variance |
| τ | Time difference $t_2 - t_1$ |
| $\Phi_{\mathbf{x}}(s)$ | Characteristic function |
| $ $ | Absolute value |
| \in | Belonging to |
| $\binom{a}{n}$ | Binomial coefficient |
| $\langle \rangle$ | Ensemble average |
| $\{ \}$ | Event |
| \cap | Intersection |