Chapter 3

Imaging of illuminated objects

§18. Presence of several luminous points

In the presence of *one* luminous surface element, the diffraction pattern is symmetrical with respect to the location of that element. This applies to an arbitrarily located surface element, as long as one limits oneself to points close to the axis of the system. *The diffraction pattern always remains stationary and moves with the luminous surface element.*

With the simultaneous presence of several luminous elements, the observed diffraction pattern depends on whether the individual elements emit independent *incoherent* waves from each other, or whether the waves emitted from individual elements are *coherent, i.e., capable of interference*.

The following laws hold, assuming that we are dealing with several luminous "*points*": *If different wave trains are incoherent, one obtains the resulting intensity at each location by simply summing the squares of the amplitudes, i.e., the intensities, that are generated by individual luminous points.*

If n luminous "points" contribute to the light disturbance at the observation point, and if the disturbance generated by their wave trains are represented by the value of the electric field (of the light vector),

$$
\begin{aligned}
\mathfrak{E}_1 &= a_1 \cos\left(2\pi\frac{t}{T} + \delta_1\right) \\
\mathfrak{E}_2 &= a_2 \cos\left(2\pi\frac{t}{T} + \delta_2\right) \\
&\ldots\ldots \\
\mathfrak{E}_n &= a_n \cos\left(2\pi\frac{t}{T} + \delta_n\right) ,
\end{aligned}
$$

then the resulting intensity in the case of *incoherent* wave trains is

$$J_{\mathrm{inc}} = \overline{\mathfrak{E}_1^2} + \overline{\mathfrak{E}_2^2} + \cdots + \overline{\mathfrak{E}_n^2} ,$$

and is within an insignificant proportionality factor $1/2$ given by

$$J_{\mathrm{inc}} = a_1^2 + a_2^2 + \cdots + a_n^2 .$$

On the other hand, if the wave trains are *coherent* and their electric field vectors $\mathfrak{E}$ have almost the same direction, which we assume for the sake of simplicity, then one has to first add the individual fields at the observation point to yield

$$\mathfrak{E} = \mathfrak{E}_1 + \mathfrak{E}_2 + \cdots + \mathfrak{E}_n .$$

The intensity is then given by

$$J_{\mathrm{coh}} = \overline{\mathfrak{E}^2} .$$

If we bring $\mathfrak{E}$ after summation into the form

$$\mathfrak{E} = A \cos 2\pi\frac{t}{T} + B \sin 2\pi\frac{t}{T} ,$$

the intensity is therefore, to within a factor of $1/2$,[xliv]

$$J_{\mathrm{coh}} = A^2 + B^2 .$$

The difference in intensity calculation for both cases is most striking for the observation point that is reached by all wave trains with the same phase. Then we have

$$J_{\text{inc}} = a_1^2 + a_2^2 + \cdots + a_n^2 \tag{31}$$

in contrast to

$$J_{\text{coh}} = (a_1 + a_2 + \cdots + a_n)^2 \,. \tag{32}$$

If additionally the amplitudes of the individual waves are of equal magnitude (a), we have

$$J_{\text{inc}} = n \cdot a^2 \,,$$
$$J_{\text{coh}} = n^2 a^2 = n \cdot J_{\text{inc}} \,.$$

If $J_{\text{coh}} > J_{\text{inc}}$ for one observation point, then there must necessarily be another point for which the wave trains do *not* arrive with the same phase, and we have $J_{\text{coh}} < J_{\text{inc}}$. *This, however, is simply the nature of interference.*

§19. Presence of several luminous surface elements

In reality, we do not deal with luminous points but surface elements. We want to represent the disturbance caused by a luminous surface element df at observation point P by the previously used auxiliary vector s that is proportional to the electric field, giving us the intensity via form $\overline{s^2}\,df$. Let

$$s_P = a\cos\left(2\pi\frac{t}{T} + \delta\right) \,,$$

where we assume that all wave trains originating from the surface element have combined physically at the location of the observation point to a single wave train with amplitude a and phase $\left(2\pi\frac{t}{T} + \delta\right)$.

An extended luminous *surface* consists of many surface elements. The calculation of intensity at the observation point must therefore also be executed differently in the presence of a luminous surface, depending on whether the constituent surface elements emit coherent or incoherent wave trains. In the case of incoherence, the intensity is simply

$$J_{\mathrm{inc}} = \int \overline{s^2}\, df$$

or to within a factor

$$J_{\mathrm{inc}} = \int a^2\, df \ , \tag{33}$$

where the integration extends over the luminous surface. If a is equal for all surface elements, then we have

$$J_{\mathrm{inc}} = a^2 \int df = a^2 F \, , \tag{33a}$$

where F is the size of the surface. In the case of coherence, on the other hand, one has to first calculate according to Huygens' principle the induced disturbance over the entire luminous surface at the observation point, that is, to form

$$S = \int \cos\left(2\pi \frac{t}{T} + \delta\right) df \ , \tag{34}$$

where again the integration extends over the luminous surface. Hereupon, one has to bring S into the canonical form

$$S = A \cos 2\pi \frac{t}{T} + B \sin 2\pi \frac{t}{T} \, . \tag{35}$$

The intensity at the observation point is then

$$J_{\mathrm{coh}} = A^2 + B^2 \, . \tag{36}$$

If there exists an observation point at which all wave trains arrive with equal phase and amplitude, then we get

$$S = a \cos\left(2\pi \frac{t}{T} + \delta\right) \int df = a \cos\left(2\pi \frac{t}{T} + \delta\right) F \, ,$$

where $\delta = \text{const}$. If we bring S to the form of Eq. 35, we have

$$A = aF\cos\delta$$
$$B = aF\sin\delta$$

and the intensity is

$$J_{coh} = a^2 \cdot F^2 \,. \tag{36a}$$

§20. Single luminous slit

Let the slit run parallel to the y-axis (vertically) and extend from $Y = -b$ to $Y = +b$; let its width be small compared to its height and therefore be designated as dX.

I. If the slit is covered with *self-luminous* surface elements, we are dealing with incoherent wave trains. The intensity at the location of the resulting diffraction pattern is to be calculated according to Eq. 33 and becomes, if one substitutes x' with $x - X$ and y' with $y - Y$, according to Eq. 30,

$$J_{inc} = \left(\frac{k}{\lambda}4\alpha'\beta'\right)^2 dX \left(\frac{\sin 2\pi\frac{(x-X)\alpha'}{\lambda}}{2\pi\frac{(x-X)\alpha'}{\lambda}}\right)^2 \int\limits_{Y=-b}^{Y=+b} dY \left(\frac{\sin 2\pi\frac{(y-Y)\beta'}{\lambda}}{2\pi\frac{(y-Y)\beta'}{\lambda}}\right)^2 . \tag{37}$$

If we set

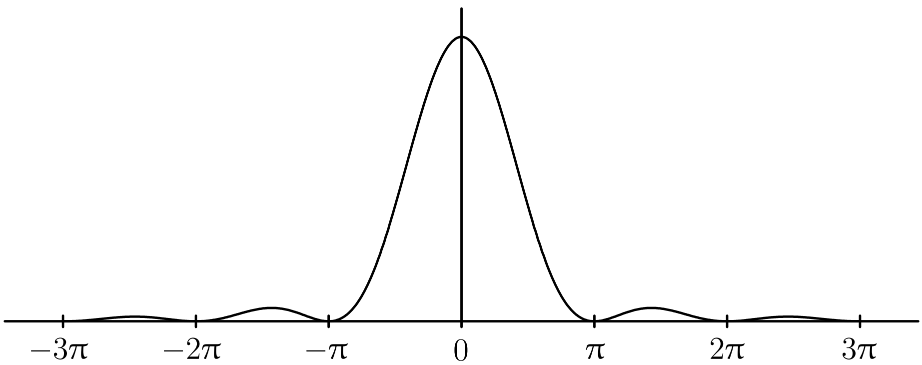
$$\frac{2\pi(y-Y)\beta'}{\lambda} = w\,,$$

then the integral appearing in Eq. 37 becomes

$$-\frac{\lambda}{2\pi\beta'} \int\limits_{2\pi(y+b)\frac{\beta'}{\lambda}}^{2\pi(y-b)\frac{\beta'}{\lambda}} \left(\frac{\sin w}{w}\right)^2 dw = +\frac{\lambda}{2\pi\beta'} \int\limits_{2\pi(y-b)\frac{\beta'}{\lambda}}^{2\pi(y+b)\frac{\beta'}{\lambda}} \left(\frac{\sin w}{w}\right)^2 dw \,.$$

The graph of the function $\left(\frac{\sin w}{w}\right)^2$ is shown schematically in Fig. 30. The function becomes zero at the same locations as

Figure 30



the function $\frac{\sin w}{w}$ that was previously discussed in more detail; the greatest maximum of w, having a value of one, also lies at $w = 0$, whereas the secondary maxima are consistently smaller than those of the function $\frac{\sin w}{w}$, and the entire curve lies above the w-axis because of its quadratic character.

The integral is represented by the areal content between the w-axis and the segment of the curve that is cut out by lines

$$w_1 = 2\pi(y - b)\beta'/\lambda$$

and

$$w_2 = 2\pi(y + b)\beta'/\lambda \,.$$

The limits of the integral are different depending on the location of the observation point xy relative to the luminous slit. If we define as "*slit zone*" the areal strip formed by moving the slit parallel to itself in both directions of the x-axis, we can distinguish three cases: the observation point lies outside the slit zone, in the immediate vicinity of its borders, or within the slit zone.

1. $y > +b$ or $y < -b$ and $|y - b|$ is large compared to $\lambda' = \lambda/\beta'$; i.e., the observation point lies a considerable number of wavelengths away from the edges to the outside. Then we can set both limits of the integral to infinity, in fact both positive if $y > b$ and both negative if $y < b$. The integral here becomes negligibly small.
2. $y = \pm b$: In this case, the limits of the integral become 0 and ∞ or ∞ and 0, and the integral itself takes on the value $\pi/2$ since we know[xlv]

$$\int_0^\infty \left(\frac{\sin w}{w}\right)^2 dw = \pi/2\,. \tag{38}$$

3. $y < b$ and $y > -b$ and further $|b - y|$ large compared to $\lambda' = \lambda/\beta'$; i.e., the observation point lies within the slit zone, but a considerable number of wavelengths away from the edges. In this case we can replace the limits of the integral by $-\infty$ and $+\infty$, and the integral takes on the value of π.

For the intensity in Eq. 37, the integral under consideration is multiplied by a function of x; accordingly, the intensity of light is zero for all points outside the slit zone (case 1). For points in the slit zone and near the borders (cases 3 and 2), the intensity depends only on x and drops suddenly to half the value if the observation point moves for constant x into one of the edges of the slit zone.

The intensity in the direction along the x-axis is given by the expression

$$J_{\mathrm{inc}} = C \cdot \left(\frac{k}{\lambda} 4\alpha'\beta'\right)^2 dX \cdot \frac{\lambda}{2\pi\beta'} \left(\frac{\sin 2\pi \frac{(x-X)\alpha'}{\lambda}}{2\pi \frac{(x-X)\alpha'}{\lambda}}\right)^2 , \qquad (39)$$

where $C = 0$ for case 1, $C = \pi/2$ for case 2, and $C = \pi$ for case 3. This functional dependence is, apart from a constant factor, the one schematically drawn in Fig. 30.

II. If the slit is covered with *illuminated* (i.e., not self-luminous) surface elements, then we are dealing with *coherent* wave trains. We therefore have to calculate the intensity according to Eqs. 34, 35, and 36, so that we obtain

$$J_{\mathrm{coh}} = \left[\frac{k}{\lambda} 4\alpha'\beta' \, dX \frac{\sin 2\pi \frac{(x-X)\alpha'}{\lambda}}{2\pi \frac{(x-X)\alpha'}{\lambda}} \int_{-b}^{+b} dY \cdot \frac{\sin 2\pi \frac{(y-Y)\beta'}{\lambda}}{2\pi \frac{(y-Y)\beta'}{\lambda}}\right]^2 ; \quad (40)$$

if we set $\frac{2\pi(y-Y)\beta'}{\lambda} = w$, the integral becomes

$$+\frac{\lambda}{2\pi\beta'} \int_{2\pi \frac{y-b}{\lambda'}}^{2\pi \frac{y+b}{\lambda'}} \frac{\sin w}{w} \, dw \; .$$

The function $\frac{\sin w}{w}$ has the graph drawn in Fig. 27. Since the curve lies partly below the w-axis, the sign of the areal patches represented by the integral changes, and the value of the integral therefore approaches a finite limit as w increases, faster than the integral in case I, all else being equal.

To find the intensity versus position, we have to consider as well the three cases separately, where the observation point is inside, outside, and on the edges of the slit zone. Since this integral is

$$\int_{-\infty}^{+\infty} \frac{\sin w}{w} \, \mathrm{d}w = \pi \tag{41}$$

again, the resulting diffraction pattern has exactly the same appearance as in the case of the self-luminous slit. For homologous points the intensity differs only by a constant factor, and at the edges of the slit zone it goes to zero via the half-value even faster for the illuminated slit than in the case of the self-luminous slit.

§21. Two parallel and neighboring slits

Each of the two slits shall again be assumed to be *infinitely narrow*. Let their distance Δ be finite but of arbitrary value. As before, we would like to treat the case of two self-luminous slits separately from the case in which the slits receive their light from an external source. In the latter case, we also need to discuss the influence on the diffraction pattern exerted by the position of the light source on the illuminated slits. This is because only with oblique illumination do noticeable differences between diffraction patterns of self-luminous and illuminated double slits become evident.

I. *Self-luminous slits.* Each slit generates the diffraction pattern that was discussed in § 20 under I, whose appearance is completely identical for both slits. The center of each individual diffraction pattern coincides with the center of the slit that generates it. Thus, we are dealing with the superposition of two identical diffraction patterns whose principal maxima are separated from each other in the direction of the x-axis by the distance Δ of the two light slits. Since we are dealing with a self-luminous double slit, the resulting intensity at each location is therefore the sum of the intensities caused by each luminous slit.

This expression is given by the formula

$$J_{\text{inc}} = \text{const.}\left[\frac{\sin 2\pi\frac{(x-X_1)\alpha'}{\lambda}}{2\pi\frac{(x-X_1)\alpha'}{\lambda}}\right]^2 + \text{const.}\left[\frac{\sin 2\pi\frac{[x-(X_1+\Delta)]\alpha'}{\lambda}}{2\pi\frac{[x-(X_1+\Delta)]\alpha'}{\lambda}}\right]^2, \tag{42}$$

where X_1 is the abscissa of the first luminous slit and $X_1 + \Delta$ is the abscissa of the second luminous slit.

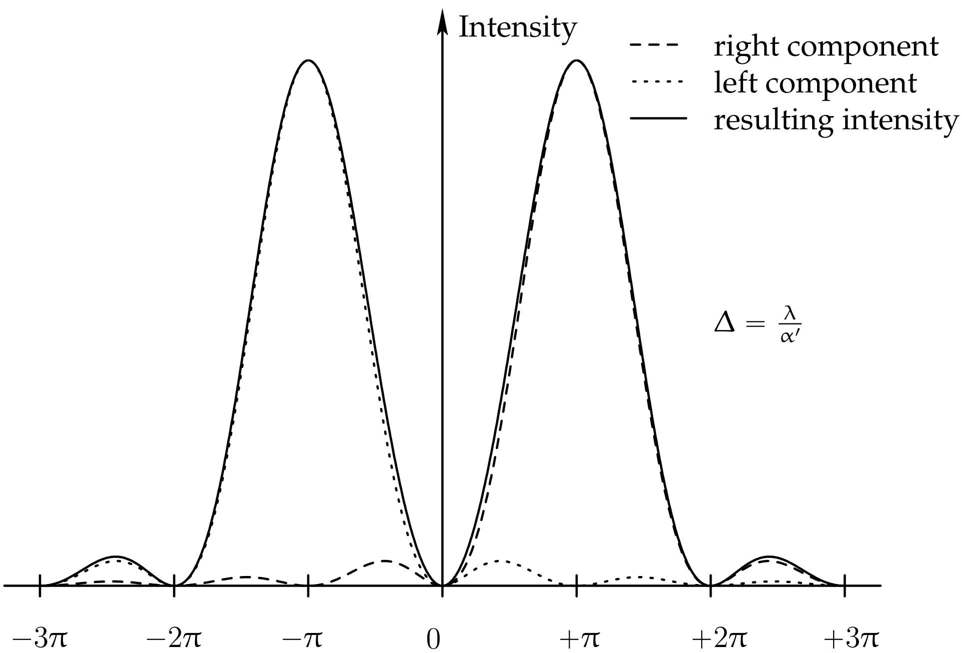
We want to carry out the discussion of this expression only for the two special cases $\Delta = \lambda/\alpha'$ and $\Delta = \lambda/2\alpha'$.

1. $\Delta = \lambda/\alpha'$. Then the expression for the resulting intensity becomes

$$J_{\text{inc}} = \text{const.}\left[\frac{\sin 2\pi\frac{x-X_1}{\Delta}}{2\pi\frac{x-X_1}{\Delta}}\right]^2 + \text{const.}\left[\frac{\sin\left(2\pi\frac{x-X_1}{\Delta} - 2\pi\right)}{2\pi\frac{x-X_1}{\Delta} - 2\pi}\right]^2.$$

We recognize easily that the two intensity curves are simply shifted along the x-axis by a distance 2π (Fig. 31). Each of the principal maxima coincides with the second minimum of the other curve, while the first minima coincide and bisect the distance $\Delta = \lambda/\alpha'$. By summing the ordinates we obtain the resulting intensity curve, which is shown as the solid line in the figure. This curve exhibits two principal maxima separated by the distance of the two luminous slits ($\Delta = \lambda/\alpha'$), and a steady and symmetrical decrease in brightness that reaches the value zero in the middle between the principal maxima. Going outward on both sides there is a series of secondary maxima that are separated from each other by complete minima.

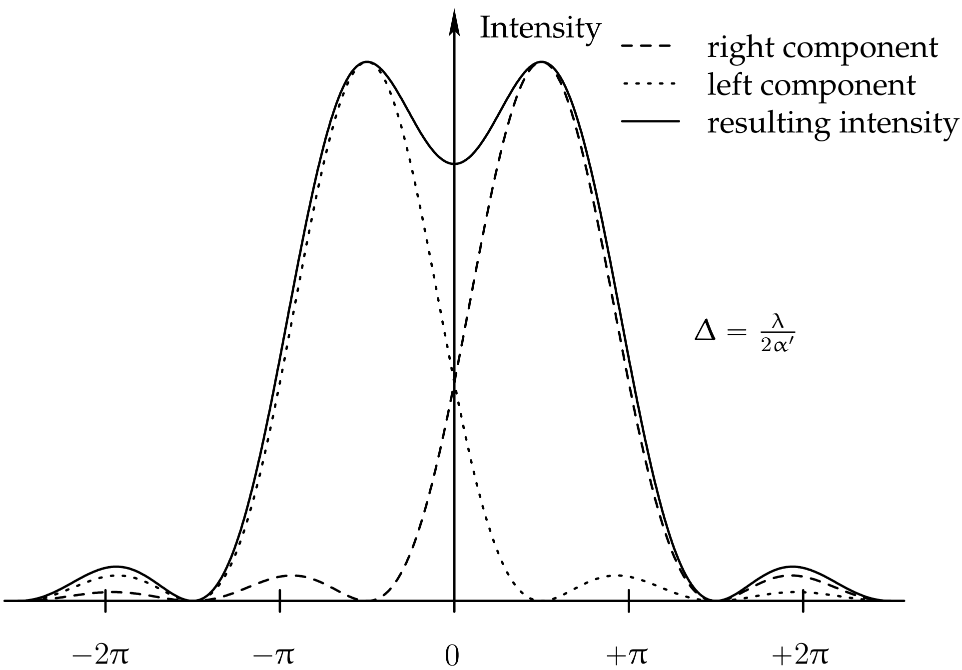
Figure 31



2. $\Delta = \lambda/2\alpha'$. In this case, an analogous observation shows that by superposing the two intensity curves, both principal maxima merge into a single, correspondingly wider central strip that exhibits a small intensity attenuation in the center. The first secondary maxima are still clearly noticeable (Fig. 32).

II. *Illuminated slits.* In this case, we are dealing with two infinitely narrow slits of finite separation that receive their light from an external source. As such, we would like to consider the intensely bright filament of a light bulb that is located in the focal plane of an objective lens, so that *plane* waves are emitted.

Figure 32



Let the filament be parallel to the direction of the slits. If the axis of this collimator is perpendicular to the plane of the slits, then coherent wave trains are emitted from there with zero phase difference. Their phase difference deviates from zero, however, if the axis of the collimator is tilted with respect to the plane of the slits.

We first consider the case of normal incidence. If we designate the angle of incidence of the light rays by u, then this case is characterized by $u = 0$.

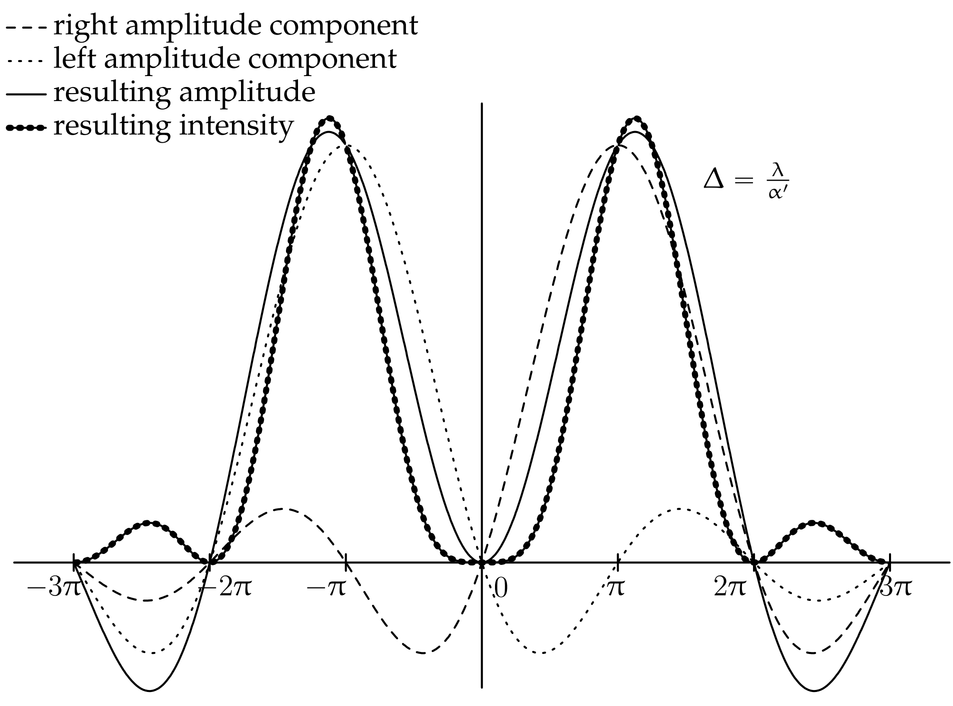
A. $u = 0$. The resulting intensity in the case of coherent wave trains is given by the expression

$$J_{coh} = \left[\text{const.}\frac{\sin 2\pi\frac{(x-X_1)\alpha'}{\lambda}}{2\pi\frac{(x-X_1)\alpha'}{\lambda}} + \text{const.}\frac{\sin 2\pi\frac{[x-(X_1+\Delta)]\alpha'}{\lambda}}{2\pi\frac{[x-(X_1+\Delta)]\alpha'}{\lambda}} \right]^2, \tag{43}$$

where the previous designations are kept. For this case, too, we would like to discuss in more detail this expression for the two special cases $\Delta = \lambda/\alpha'$ and $\Delta = \lambda/2\alpha'$.

1. $\Delta = \lambda/\alpha'$. In this case, both amplitude curves are shifted from each other by 2π in the direction of the x-axis and drawn in Fig. 33.

Figure 33



By summing the ordinates algebraically one obtains the resulting amplitude, and by squaring it one obtains the intensity of the resulting diffraction pattern. One can easily see that the two principal maxima are separated by a perfect minimum. The decrease of intensity toward this minimum is happening here more rapidly than in the analogous case of self-luminous slits. Going outward, the principal maxima are followed once again by secondary maxima, which in turn are separated from each other by perfect minima. The intensities of the corresponding secondary maxima are of greater magnitude than in the former case.

2. $\Delta = \lambda/2\alpha'$. For this case, Fig. 34 shows the respective position of the two amplitude curves. A consideration analogous to the above teaches us that the two principal maxima again merge into a single bright central strip that, in contrast to the analogous case of self-luminous slits, is brighter and drops faster, whereas, conversely, the secondary maxima are evidently much weaker than the former.

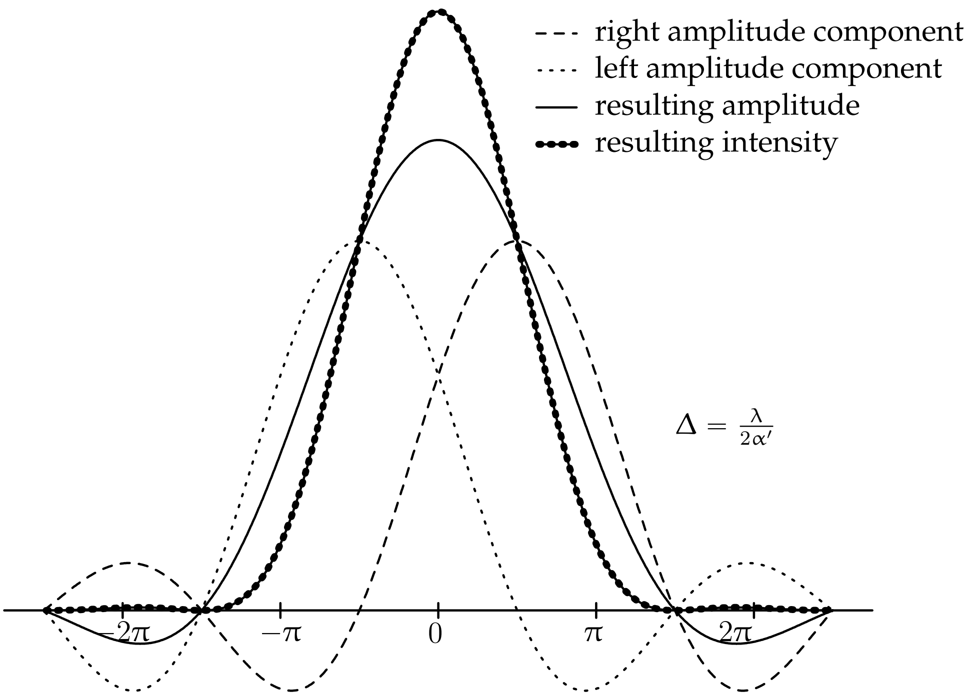
B. Angle of incidence $u > 0$. In Fig. 35, let Sl_1 and Sl_2 be the locations of the two slits of separation Δ, which are met by light at an angle u. As before, let the slits be so narrow that the phase can be considered constant even under oblique incidence of light. The path difference for them is therefore

$$\Delta \sin u \,,$$

so that the coherent disturbances emanating from Sl_1 and Sl_2 can be represented by

$$\alpha \sin 2\pi \left(\frac{t}{T} + \frac{1}{2} \frac{\Delta \sin u}{\lambda} \right)$$

Figure 34



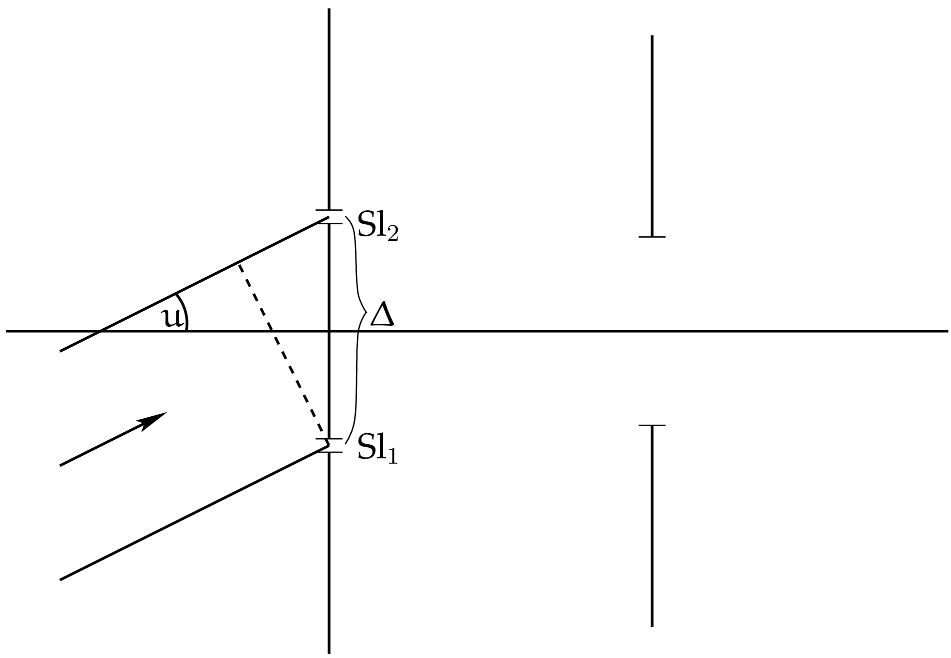
and

$$\alpha \sin 2\pi \left(\frac{t}{T} - \frac{1}{2} \frac{\Delta \sin u}{\lambda} \right) \ .$$

If only slit Sl_1 is present, then, according to earlier explanations, the disturbance at the observation point is

$$s_1 = \text{const} \frac{\sin w_1}{w_1} \sin 2\pi \left(\frac{t}{T} + \frac{1}{2} \frac{\Delta \sin u}{\lambda} \right) \ .$$

Figure 35



If only the slit Sl_2 is present, then the light disturbance at the same observation point is

$$s_2 = \mathrm{const}\frac{\sin w_2}{w_2}\sin 2\pi\left(\frac{t}{T} - \frac{1}{2}\frac{\Delta\sin u}{\lambda}\right) .$$

The values of w_1 and w_2 are the same as those in the previously treated case of perpendicular incident light, into which our present case transitions when $u = 0$. Thus

$$\begin{cases} w_1 = \frac{2\pi(x-X_1)\alpha'}{\lambda} \\ w_2 = \frac{2\pi[x-(X_1+\Delta)]\alpha'}{\lambda} \end{cases} .$$

If both slits act simultaneously, the light disturbance at the observation point is then given by

$$\begin{aligned} S = s_1 + s_2 &= \text{const}\left(\frac{\sin w_1}{w_1} + \frac{\sin w_2}{w_2}\right)\cos\pi\frac{\Delta\sin u}{\lambda}\sin 2\pi\frac{t}{T} \\ &+ \text{const}\left(\frac{\sin w_1}{w_1} - \frac{\sin w_2}{w_2}\right)\sin\pi\frac{\Delta\sin u}{\lambda}\cos 2\pi\frac{t}{T} \\ &= A\sin 2\pi\frac{t}{T} + B\cos 2\pi\frac{t}{T}\,, \end{aligned}$$

so that the intensity becomes

$$\begin{aligned} J_{coh} = A^2 + B^2 = \text{const}^2 \Bigg[\left(\frac{\sin w_1}{w_1}\right)^2 + \left(\frac{\sin w_2}{w_2}\right)^2 \\ +2\frac{\sin w_1}{w_1}\frac{\sin w_2}{w_2}\cos 2\pi\frac{\Delta\sin u}{\lambda}\Bigg]. \end{aligned} \tag{44}$$

It is readily apparent that this expression becomes identical with that for two *self*-luminous slits of equal separation Δ (see § 21, I) in case the cosine disappears. This is the case for

$$\frac{2\pi\Delta\sin u}{\lambda} = \pm(2a+1)\frac{\pi}{2}\,, \qquad a = 0, 1, 2,$$

i.e., for

$$\sin u = \pm\frac{(2a+1)\lambda}{4\Delta}\,.$$

We further see that the expression assumes likewise a very simple form if the cosine becomes $+1$ or -1. The former occurs for

$$\frac{2\pi\Delta\sin u}{\lambda} = \pm 2a\pi\,, \qquad a = 0, 1, 2,$$

i.e., for

$$\sin \mathfrak{u} = \pm \frac{\mathfrak{a}\lambda}{\Delta} \, .$$

We then have

$$J_{\mathrm{coh}} = \mathrm{const}^2 \left[\frac{\sin w_1}{w_1} + \frac{\sin w_2}{w_2} \right]^2 \, . \tag{45}$$

This intensity distribution, occurring periodically with variation of only $\mathfrak{u}$, is thus identical to that for two illuminated slits of the same separation Δ for normal incidence ($\mathfrak{u} = 0$).

The cosine becomes -1 for

$$\frac{2\pi\Delta \sin \mathfrak{u}}{\lambda} = \pm(2\mathfrak{a} + 1)\pi \, , \; \mathfrak{a} = 0, 1, 2,$$

i.e., for

$$\sin \mathfrak{u} = \pm \frac{(2\mathfrak{a} + 1)\lambda}{2\Delta} \, .$$

In this case, we have

$$J_{\mathrm{coh}} = \mathrm{const}^2 \left[\frac{\sin w_1}{w_1} - \frac{\sin w_2}{w_2} \right]^2 \, . \tag{46}$$

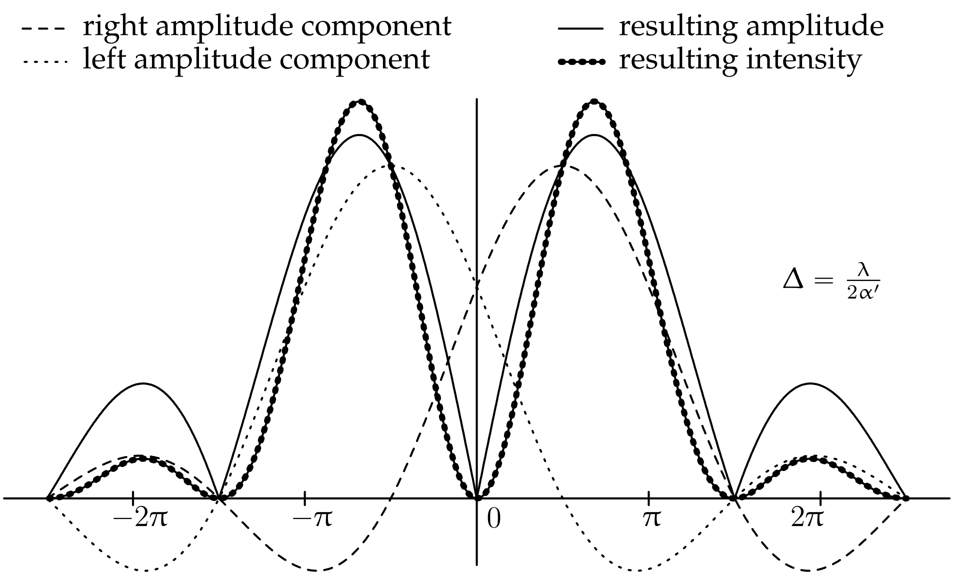
Whereas in the previous cases of coherent waves the resulting disturbance was obtained by adding the amplitudes, here the interesting case arises that the amplitudes of the individual fields are to be *subtracted* in order to obtain the resulting disturbance.

One consequence of this is the particularly noticeable difference, at these angles of incidence of light, between the diffraction pattern of self-luminous and illuminated slits of equal separation. This difference appears particularly striking for the special case $\Delta = \frac{\lambda}{2\alpha'}$ in which the angle of incidence must be

$$\sin u = (2\mathfrak{a} + 1)\alpha' .$$

For this case, Fig. 36 shows the respective position of the two amplitude curves. The resulting amplitude of the diffraction pattern is represented by the solidly drawn curve.[xlvi] It can be seen that the two principal maxima are separated by a perfect minimum, whereas in the self-luminous slits and also in the illuminated slit with normal incidence, the principal maxima are merged into a single and correspondingly broader bright central strip.

Figure 36



§22. An illuminated slit of finite width

If the slit is self-luminous, the result is easy to assess. The slit of finite width can be thought of as the result of shifting an infinitely narrow slit parallel to the x-axis. One therefore only needs to construct the diffraction pattern corresponding to the infinitely narrow slit situated at different positions and then *add the individual intensities* at each location. With the broadening of the self-luminous slit, the diffraction pattern of an infinitely narrow slit must become more and more unclear.[1]

Much more diverse are the phenomena of an *illuminated* slit of finite width. The term that gives the light disturbance at the observation point, in the case of an illuminated slit, is

$$s = \frac{k}{\lambda} \int_{-a}^{+a} \int_{-b}^{+b} 4\alpha'\beta' \, dX \, dY \, \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \cdot \frac{\sin 2\pi\beta' \frac{y-Y}{\lambda}}{2\pi\beta' \frac{y-Y}{\lambda}} \cdot \sin 2\pi \frac{t}{T}, \tag{47}$$

where $2b$ and $2a$ denote the height and width of the illuminated slit. If the slit is infinitely narrow, the integration over dX becomes unnecessary and the integrand moves as a constant to the front of the integral, a case that has already been dealt with in § 20. For an infinitely narrow slit, the position of the light source, i.e., the direction of the angle of incidence of light, is of no influence on the diffraction pattern. In the case of a finite width of the slit, on the other hand, the

[1]This is the typical difference between a diffraction phenomenon and a pure interference phenomenon with a self-luminous slit (Lummer–Haidinger interference curves of equal inclination), in which only the angular magnitude of the visual field grows with the broadening of the light source (slit).

oblique incidence of the rays brings about phase differences along dX, so that in this more general case the light disturbance becomes

$$\left.\begin{aligned} s = & \frac{k}{\lambda} \int\limits_{-a}^{+a} \int\limits_{-b}^{+b} 4\alpha'\beta'\, dX\, dY\, \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \\ & \cdot \frac{\sin 2\pi\beta' \frac{y-Y}{\lambda}}{2\pi\beta' \frac{y-Y}{\lambda}} \cdot \sin 2\pi \left(\frac{t}{T} - \frac{X \sin u}{\lambda} \right) \end{aligned}\right\}, \tag{48}$$

where u is the angle of incidence of the incoming plane wave.

This expression can be written in the following form:

$$\left.\begin{aligned} s = & \frac{k}{\lambda} \int\limits_{-b}^{+b} dY\, 2\beta' \frac{\sin 2\pi\beta' \frac{y-Y}{\lambda}}{2\pi\beta' \frac{y-Y}{\lambda}} \\ & \cdot \int\limits_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \cdot \sin 2\pi \left(\frac{t}{T} - \frac{X \sin u}{\lambda} \right) \end{aligned}\right\}. \tag{49}$$

This form reflects the formation of the resulting disturbance at the observation point. Consider the slit as a checkered pattern consisting of individual surface elements of size $dX\, dY$; the above form, when calculating the disturbance at the observation point, initially takes into account only the influence of surface elements located on a strip parallel to the y-axis with width dX and height $2b$, so that the first integral in itself represents the already treated case of an infinitely narrow illuminated slit. As we know, the value of this integral is, to within a constant, equal to π for observation points within the "slit zone" (see § 20).

The slit of finite width may be assembled purely from such strips whose effect at the observation point is a function of the location of the single strip and the prevailing phase there; i.e., it is a function of

X. This influence of the width is taken into account by the second integral.

In the calculation of s, we first restrict ourselves to the case in which the phase is the same for all individual strips, i.e., we assume normal incidence of light ($u = 0$). Then we have to consider the following integral:

$$J = \int_{-a}^{+a} dX \, \frac{\sin 2\pi\alpha' \frac{(x-X)}{\lambda}}{2\pi\alpha' \frac{(x-X)}{\lambda}} \ .$$

To solve this integral, we employ an artifice. It is known that

$$\frac{\sin 2\pi\alpha'\mu}{\pi\mu} = \int_{-\alpha'}^{+\alpha'} \cos(2\pi\mu\nu) \, d\nu \ .$$

So if we set

$$\mu = \frac{x - X}{\lambda} \ ,$$

the integral becomes

$$J = \frac{1}{2\alpha'} \int_{-a}^{+a} dX \int_{-\alpha'}^{+\alpha'} \cos\left(2\pi\nu \frac{x - X}{\lambda}\right) d\nu \ ,$$

and by switching the order of integration,

$$J = \frac{1}{2\alpha'} \int_{-\alpha'}^{+\alpha'} d\nu \int_{-a}^{+a} dX \cos\left(2\pi\nu \frac{x - X}{\lambda}\right) \ .$$

Now we can carry out the integration over X and get

$$J = \frac{1}{2\alpha'} \int_{-\alpha'}^{+\alpha'} d\nu \, \frac{\sin 2\pi\nu\frac{x+a}{\lambda} - \sin 2\pi\nu\frac{x-a}{\lambda}}{2\pi\frac{\nu}{\lambda}}$$
$$= \int_{-\alpha'}^{+\alpha'} d\nu \, \frac{\cos 2\pi\nu\frac{x}{\lambda} \cdot \sin 2\pi\nu\frac{a}{\lambda}}{2\pi\frac{\nu}{\lambda}\alpha'} \; .$$

If we set

$$2\pi\nu\frac{a}{\lambda} = w \; ,$$

then we obtain

$$J = \frac{\lambda}{2\pi\alpha'} \int_{-2\pi a\frac{\alpha'}{\lambda}}^{+2\pi a\frac{\alpha'}{\lambda}} dw \, \frac{\cos\left(\frac{x}{a}w\right)\sin w}{w} \; . \tag{50}$$

We can see that this integral is a function of x; we would like to compare it with the integral

$$J_0 = \frac{1}{\pi} \int_{-\infty}^{+\infty} dw \, \frac{\cos\left(\frac{x}{a}w\right)\sin w}{w} \; . \tag{51}$$

To find the value of the integral in Eq. 51, we start with the task of determining a function of x such that it takes on the value of 1 between $x = -a$ and $x = +a$, and the value 0 everywhere else.

In general, according to the Fourier integral theorem,[xlvii]

$$f(x) = \frac{1}{\pi} \int_0^{\infty} dz \int_{-\infty}^{+\infty} f(u) \cos z(u - x) \, du \; . \tag{52}$$

The function that we seek is therefore

$$f(x) = \frac{1}{\pi}\int\limits_0^{\infty} dz \int\limits_{-a}^{+a} \cos z(u-x)\,du$$
$$= \frac{1}{\pi}\int\limits_0^{\infty} dz\,\frac{2}{z}\sin(az)\cos(zx)\;,$$

or, if we set additionally $az = w$,

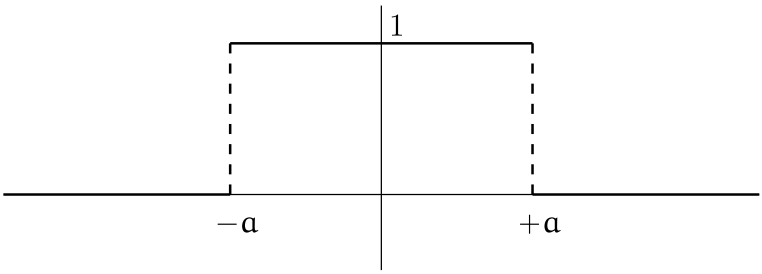
$$f(x) = \frac{2}{\pi}\int\limits_0^{\infty} dw\,\frac{\sin w \cos\left(\frac{x}{a}w\right)}{w}$$
$$= \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dw\,\frac{\sin w \cos\left(\frac{x}{a}w\right)}{w} = J_0\;.$$

The value of J_0 as a function of x is therefore

$$\left.\begin{array}{l} J_0 = 0 \text{ for } \begin{cases} x > -\infty \text{ and } < -a \\ x > +a \end{cases} \\ J_0 = 1 \text{ for } \begin{cases} x > -a \text{ and} \\ x < +a \end{cases} \\ J_0 = \dfrac{1}{2} \text{ for } \begin{cases} x = +a \\ x = -a \end{cases} \end{array}\right\}\;. \tag{53}$$

Its graph is represented by solid lines in Fig. 37. If $a\alpha'$ is much greater than λ, then $J = \frac{\lambda}{2\alpha'} J_0$ and the light distribution in the resulting diffraction pattern is a *uniformly* bright strip of width $2a$, outside of which there is complete darkness. This light distribution in the image becomes all the more congruent to that of the object (the illuminated

Figure 37



slit), the greater the width a for a given opening angle α' of the diffracting aperture or the larger the opening angle for a given slit width.

To gain an overview as to what values of the limits permit the use of the integral J_0 instead of the integral J, we consider the following:

$$\int\limits_{-2\pi\frac{a\alpha'}{\lambda}}^{+2\pi\frac{a\alpha'}{\lambda}} = \int\limits_{-\infty}^{+\infty} - \int\limits_{-\infty}^{-2\pi\frac{a\alpha'}{\lambda}} - \int\limits_{+2\pi\frac{a\alpha'}{\lambda}}^{+\infty} .$$

Since the function to be integrated is an even function, the last two integrals on the right are the same and we can write

$$J = \frac{\lambda}{2\alpha'} J_0 - \frac{\lambda}{\pi\alpha'} \int\limits_{\frac{2\pi a\alpha'}{\lambda}}^{\infty} dw \, \frac{\sin w \cos\left(\frac{x}{a} w\right)}{w} , \tag{54}$$

so that the amplitude of the resulting disturbance becomes

$$\text{const} \left\{ J_0 - \frac{2}{\pi} \int\limits_{\frac{2\pi a\alpha'}{\lambda}}^{\infty} dw \, \frac{\sin w}{w} \cos\left(\frac{x}{a} w\right) \right\} .$$

The integrand of the residual integral differs from the previously discussed $\left(\frac{\sin w}{w}\right)$ only by a factor $\cos\left(\frac{x}{a}w\right)$, which takes on the maximum value one. The residual integral is therefore, for all values of x, less than the integral

$$\mathfrak{P} = \frac{\lambda}{\alpha'\pi} \int\limits_{\frac{2\pi a\alpha'}{\lambda}}^{+\infty} dw \, \frac{\sin w}{w} \, .$$

If, for certain values of $2\pi\frac{a\alpha'}{\lambda}$, this integral is negligible with respect to the same integral between $-\infty$ and $+\infty$, then we have a stronger reason to neglect our residual integral in comparison to J_0. The following table shows the values of the integral as a function of its lower limit $2\pi\frac{a\alpha'}{\lambda}$:

$2\pi\frac{a\alpha'}{\lambda}$	$\frac{\alpha'\pi}{\lambda}\mathfrak{P}$	$2\pi\frac{a\alpha'}{\lambda}$	$\frac{\alpha'\pi}{\lambda}\mathfrak{P}$
0	1.5708	20	0.0226
1	0.6247	50	0.0192
2	−0.0346	100	0.0086
5	0.0209	200	0.0024
10	−0.0875	500	−0.0018

It can be seen from the table that $\mathfrak{P}$ decreases very rapidly and is practically zero for a value of $2\pi\frac{a\alpha'}{\lambda} = 2$.

If, for example, half the opening angle is equal to 3°, so that α' becomes approximately equal to $1/20$, then the lower limit of $\mathfrak{P}$ equals $\pi a/10\lambda$; further, if $a = 666\lambda$, or equal to $4\,\mathrm{mm}$ for a wavelength of $\lambda = 0.6\,\mu\mathrm{m}$, then $\mathfrak{P} = 0.0024 \cdot \frac{\lambda}{\alpha'\pi}$ and therefore $J = \frac{\lambda}{2\alpha'}\{J_0 - 0.0016\}$ according to Eq. 54.

We can also write the amplitude of the resulting disturbance as

$$A(x) = \text{const}\frac{2}{\pi} \int\limits_{0}^{\frac{2\pi a\alpha'}{\lambda}} dw \, \frac{\sin w}{w} \cos\left(\frac{x}{a}w\right) \, . \tag{55}$$

For $x = 0$, i.e., in the middle of the slit, the value of the amplitude is then

$$A(0) = \text{const}\frac{2}{\pi}\int_0^{\frac{2\pi a\alpha'}{\lambda}} dw\,\frac{\sin w}{w}\,,$$

which transitions to "const" for large $\frac{2\pi a\alpha'}{\lambda}$. At the edge of the slit, for $x = a$, we get

$$A(a) = \text{const}\frac{2}{\pi}\int_0^{\frac{2\pi a\alpha'}{\lambda}} dw\,\frac{\sin w\cos w}{w} = \text{const}\frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\frac{4\pi a\alpha'}{\lambda}} \frac{\sin w'}{w'}\,dw'$$

if we set $2w = w'$. For large values of $\frac{2\pi a\alpha'}{\lambda}$ this value $= 1/2$ const, or half the value at the center. In general, this simple relationship between $A(0)$ and $A(a)$ does not exist, and the values of $A(0)$ and $2\cdot A(a)$, respectively, are apparent from Figs. 38a and b (hatched).

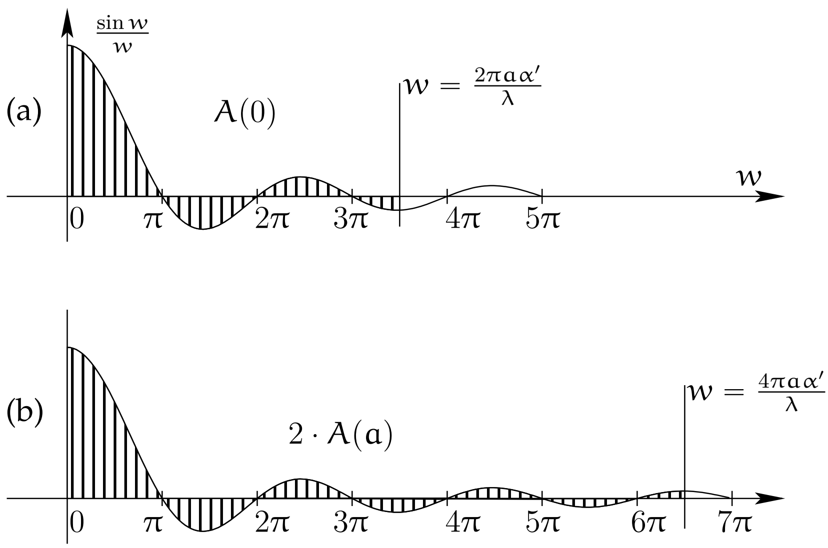
It is easy to see that in the general case, for which we cannot set $\frac{2\pi a\alpha'}{\lambda} = \infty$, the amplitude A for x inside and outside the slit fluctuates. To recognize this, we set[xlviii]

$$\begin{aligned}
\frac{dA(x)}{dx} &= \frac{d}{dx}\left\{\frac{2}{\pi}\int_0^{\frac{2\pi a\alpha'}{\lambda}} dw\,\frac{\sin w}{w}\cos\left(\frac{x}{a}w\right)\right\}\\
&= -\frac{2}{\pi}\int_0^{\frac{2\pi a\alpha'}{\lambda}} dw\,\sin w\,\sin\left(w\frac{x}{a}\right)\\
&= +\text{const}\left\{\frac{\sin u}{u} - \frac{\sin v}{v}\right\}\,,
\end{aligned}$$

where

$$u = \frac{2\pi\alpha'(a+x)}{\lambda}\,,\; v = \frac{2\pi\alpha'(a-x)}{\lambda}\,.$$

Figure 38



Let us fix, for given values of a and α', the point $\frac{2\pi\alpha' a}{\lambda}$ on the abscissa (Fig. 39), which corresponds to the point $x = 0$ (the middle of the slit), and let us go from this point to the right and left of the axis a distance $\frac{2\pi\alpha' x}{\lambda}$. Then we have in the ordinates the values of $\frac{\sin u}{u}$ and $\frac{\sin v}{v}$, whose difference is to be formed.

To fix this idea, let us choose, for example

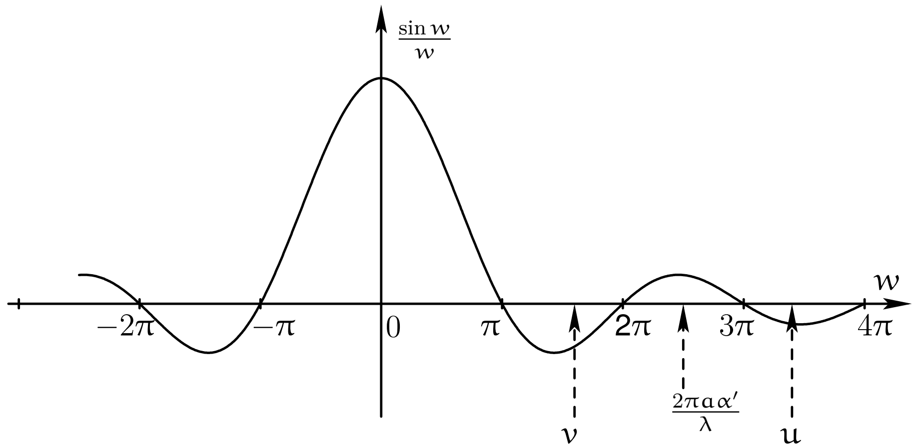
$$\frac{2\pi a\alpha'}{\lambda} = 2\pi\,,$$

so it is easy to see that if we let x grow from zero, first

$$\frac{\sin u}{u} - \frac{\sin v}{v}\,,\ \text{i.e.,}\ \frac{dA(x)}{dx}$$

is positive until it grows to a maximum value, then decays, and for $u = 3\pi$, $v = \pi$, i.e., for $x = a/2$, it is again zero. From there,

Figure 39



$\frac{dA(x)}{dx}$ becomes negative and reaches its largest negative value for $u = 4\pi$, $v = 0$, i.e., for $x = a$ at the edge of the slit. If x is allowed to grow beyond the edge of the slit, $\frac{dA(x)}{dx}$ increases again from its minimum value and reaches the value 0 for $u = 5\pi$, $v = -\pi$, i.e., $x = 3/2a$; in this way, the fluctuations of $\frac{dA(x)}{dx}$ continue and gradually die down.

Accordingly, the amplitude distribution will look somewhat like what is shown in Fig. 40.

If we choose $\frac{2\pi a\alpha'}{\lambda} = \pi$, the graph of the amplitude $A(x)$ in the interior of the slit is somewhat different; the maximum is then at $x = 0$ (see Fig. 41).

If $\frac{2\pi a\alpha'}{\lambda}$ is very small compared to π, then we can place in the expression for $A(x)$ the nearly constant factor $\frac{\sin w}{w} = 1$ in front of the integral and get

Figure 40

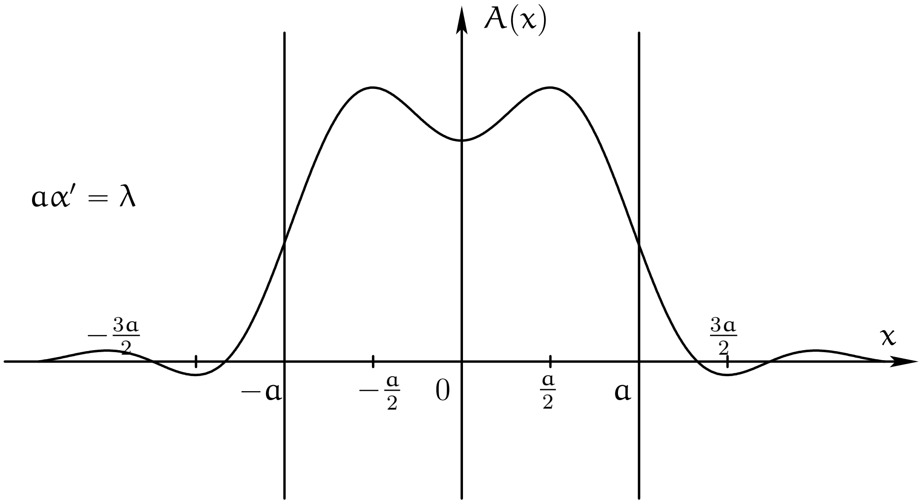
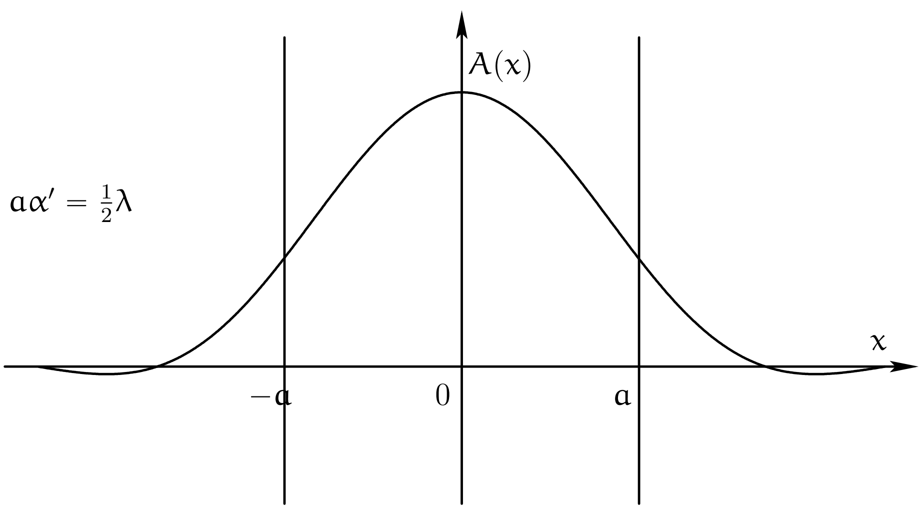


Figure 41



$$A(x) = \frac{2}{\pi} \int_0^{\frac{2\pi a\alpha'}{\lambda}} dw \cos\left(\frac{xw}{a}\right)$$
$$= \frac{2a}{\pi x} \cdot \sin\left(\frac{2\pi\alpha' x}{\lambda}\right)$$
$$= \frac{4a\alpha'}{\lambda} \cdot \frac{\sin\left(\frac{2\pi\alpha' x}{\lambda}\right)}{\frac{2\pi\alpha' x}{\lambda}} \,. \tag{56}$$

$A(x)$ has in this case the already discussed form $\frac{\sin w}{w}$. If $\frac{2\pi a\alpha'}{\lambda}$ is very large compared to π, then, as can be seen from the consideration of the form of $\frac{dA(x)}{dx}$, the fluctuations of the amplitude inside the slit are very small, and the value of the amplitude is therefore almost constant; only *at the edges* of the slit do fluctuations take place; namely (if we consider only positive values of x, since the phenomenon is symmetrical with respect to the J-axis), since $\frac{2\pi a\alpha'}{\lambda}$ was already assumed to be large, u is a fortiori large and therefore:

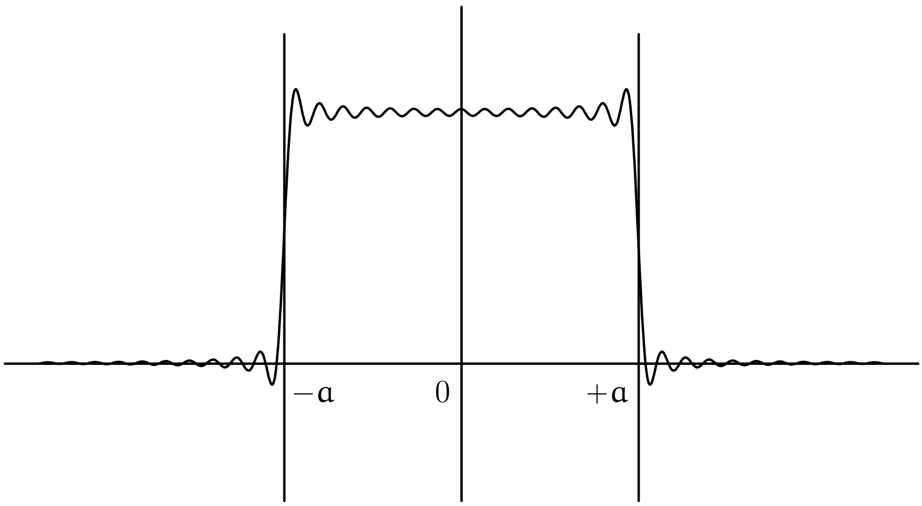
$$\frac{dA(x)}{dx} = -\text{const} \cdot \frac{\sin \nu}{\nu} \,.$$

Therefore, as ν gets closer and closer to the value $\nu = 0$ (as x increases), i.e., $x = a$ (edge of the slit), the fluctuations of $\frac{\sin\nu}{\nu}$ begin to become more and more noticeable. We therefore obtain the image of the amplitude indicated in Fig. 42:[xlix] the larger $\frac{a\alpha'}{\lambda}$ becomes, the more the variations at the edges converge, so that in the limit, for infinitely large $\frac{a\alpha'}{\lambda}$, we obtain the amplitude graph already shown in Fig. 37 above.

§23. Finite slit whose two halves possess a constant difference in phase

Let the slit have width $2a$ and height $2b$; let the phase in the half slit of height $2b$ and width a ($x = -a$ to $x = 0$) be equal to $2\pi\frac{t}{T}$, while

Figure 42



in the other half slit ($x = 0$ to $x = +a$) let it be $2\pi\frac{t}{T} + \delta$. Then the resulting light disturbance at the observation point is

$$s = \frac{k}{\lambda}\int_{-b}^{+b} dY\, 2\beta' \frac{\sin 2\pi\beta' \frac{y-Y}{\lambda}}{2\pi\beta' \frac{y-Y}{\lambda}} \left\{ \int_{-a}^{0} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \sin 2\pi\frac{t}{T} \right.$$
$$\left. + \int_{0}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \sin\left(2\pi\frac{t}{T} + \delta\right) \right\} . \tag{57}$$

If the observation point lies within the slit zone, then, as was shown previously (§ 20), the integral stretched out over dY becomes equal to λ;[1] if we split up $\sin\left(2\pi\frac{t}{T} + \delta\right)$, we get

$$s = A \sin 2\pi\frac{t}{T} + B \cos 2\pi\frac{t}{T} ,$$

where A and B are given by

$$A = k \cdot \left\{ \int\limits_{-a}^{0} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} + \cos\delta \int\limits_{0}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \right\}$$

$$B = k \sin\delta \int\limits_{0}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} .$$

To obtain the intensity, we have to form

$$\left.\begin{aligned} I &= A^2 + B^2 \\ \text{or } I &= J_1^2 + J_2^2 + 2J_1J_2 \cos\delta , \\ \text{where } J_1 &= 2\alpha' k \int\limits_{-a}^{0} dX \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \\ J_2 &= 2\alpha' k \int\limits_{0}^{a} dX \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \end{aligned}\right\} . \tag{58}$$

We want to treat more special cases.

1. If $\delta = 0, 2\pi, 4\pi$, etc., i.e., the phase difference $0, \lambda, 2\lambda$, etc., then we have $I = (J_1 + J_2)^2 = J^2$, where

$$J = 2\alpha' k \int\limits_{-a}^{+a} dX \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} ;$$

   i.e., the intensity is of the same value as if the slit had *no* phase difference.

2. If $\delta = \pi, 3\pi, 5\pi \ldots$, i.e., the phase difference is $= \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}$, etc., then we obtain

$$I = (J_1 - J_2)^2 . \tag{59}$$

If we set

$$2\pi\alpha' \cdot \frac{x - X}{\lambda} = w\,,$$

then J_1 and J_2 take on the following values:

$$\left.\begin{aligned} J_1 &= \frac{\lambda k}{\pi} \int\limits_{\frac{2\pi\alpha' x}{\lambda}}^{2\pi\alpha'\frac{x+a}{\lambda}} \frac{\sin w}{w}\, dw \\ J_2 &= \frac{\lambda k}{\pi} \int\limits_{2\pi\alpha'\frac{x-a}{\lambda}}^{\frac{2\pi\alpha' x}{\lambda}} \frac{\sin w}{w}\, dw \end{aligned}\right\} . \tag{60}$$

We see immediately that for $x = 0$, i.e., the center of the slit, $J_1 = J_2$ and therefore $I = 0$; in the middle of the slit there is a minimum, independent of the size of a.

To discuss further, we distinguish the two cases for which $\frac{2\pi\alpha' a}{\lambda}$ is small or large compared to π.

I. If $\frac{2\pi\alpha' a}{\lambda}$ is small, then we can expand according to Taylor's theorem as follows:[li]

$$\begin{aligned} J_1 =& \frac{\lambda k}{\pi} \left\{ \frac{\sin \frac{2\pi\alpha' x}{\lambda}}{\frac{2\pi\alpha' x}{\lambda}} \cdot \frac{2\pi\alpha' a}{\lambda} \right. \\ &\left. + \frac{\frac{2\pi\alpha' x}{\lambda} \cos \frac{2\pi\alpha' x}{\lambda} - \sin \frac{2\pi\alpha' x}{\lambda}}{\left(\frac{2\pi\alpha' x}{\lambda}\right)^2} \cdot \frac{\left(\frac{2\pi\alpha' a}{\lambda}\right)^2}{2!} \right\} \\ J_2 =& -\frac{\lambda k}{\pi} \left\{ \frac{\sin \frac{2\pi\alpha' x}{\lambda}}{\frac{2\pi\alpha' x}{\lambda}} \left(-\frac{2\pi\alpha' a}{\lambda} \right) \right. \\ &\left. + \frac{\frac{2\pi\alpha' x}{\lambda} \cos \frac{2\pi\alpha' x}{\lambda} - \sin \frac{2\pi\alpha' x}{\lambda}}{\left(\frac{2\pi\alpha' x}{\lambda}\right)^2} \cdot \frac{\left(-\frac{2\pi\alpha' a}{\lambda}\right)^2}{2!} \right\} , \end{aligned}$$

and get

$$J_1 - J_2 = \frac{\lambda k}{\pi} \frac{\frac{2\pi\alpha' x}{\lambda} \cos \frac{2\pi\alpha' x}{\lambda} - \sin \frac{2\pi\alpha' x}{\lambda}}{\left(\frac{2\pi\alpha' x}{\lambda}\right)^2} \cdot \left(\frac{2\pi\alpha' a}{\lambda}\right)^2 .$$

$$\text{If we set } \begin{cases} \dfrac{\lambda k}{\pi}\left(\dfrac{2\pi\alpha' a}{\lambda}\right)^2 = c\,, \\ \qquad\qquad \dfrac{2\pi\alpha' x}{\lambda} = \xi\,, \end{cases}$$

then we get

$$J_1 - J_2 = c\frac{\xi \cos \xi - \sin \xi}{\xi^2} = cf(\xi)\,.$$

To discuss the curve represented by the odd function

$$f(\xi) = \frac{\xi \cos \xi - \sin \xi}{\xi^2}\,,$$

we first determine its *zeros*. It turns out that

$$f(\xi) = 0 \text{ for } \xi = \tan \xi\,;$$

i.e., the zeros of the curve $f(\xi)$ lie at the intersections of the curves

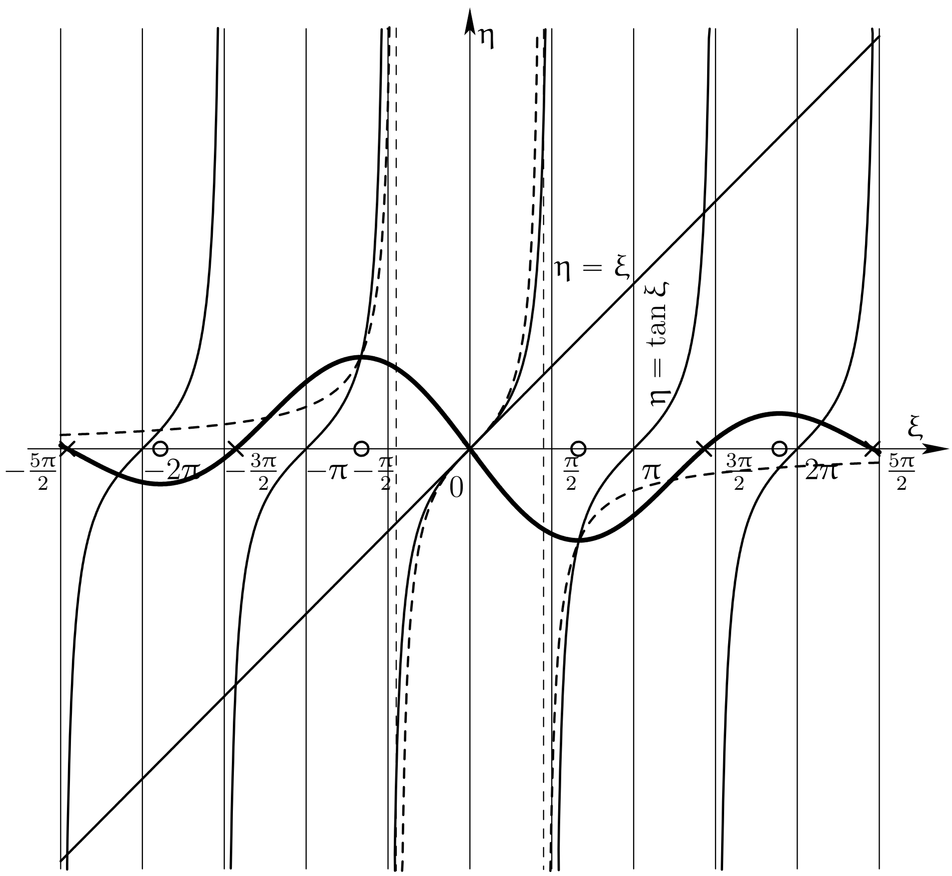
$$\eta = \xi \text{ and } \eta = \tan \xi\,.$$

The locations indicated by $\times$ in Fig. 43 are the zeros of the function $f(\xi)$; with growing $|\xi|$, the zeros thus approach the values $\pm(2a+1)\frac{\pi}{2}$ more and more closely.

We now determine the positions of the maxima and minima of $f(\xi)$. Its derivative is

$$f'(\xi) = \frac{-\xi^2 \sin \xi - 2\xi \cos \xi + 2 \sin \xi}{\xi^3}\,.$$

Figure 43



The maxima and minima of $f(\xi)$ are therefore at the locations for which

$$\tan \xi = \frac{2\xi}{2 - \xi^2} \, ,$$

i.e., at the intersections of the curves

$$\eta = \tan \xi \text{ and } \eta = \frac{2\xi}{2-\xi^2} \; .$$

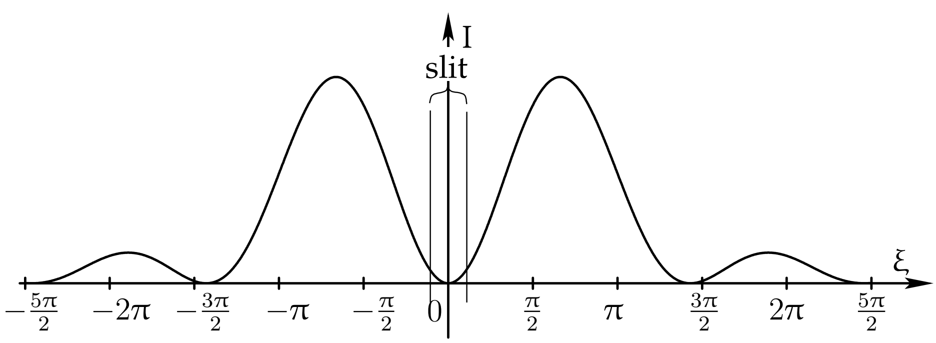
Curve $\eta = \frac{2\xi}{2-\xi^2}$ has the form that is represented by the dashed lines in Fig. 43. The points marked by $\circ$ are therefore the locations of the maxima and minima of $f(\xi)$, and $f(\xi)$ itself is approximately represented by the bold curve. $f(\xi)$ has for negative ξ opposite but equal values as for positive ξ.

If the intensity $I = c^2[f(\xi)]^2$ is formed, the intensity distribution shown in Fig. 44 is obtained. As we can see, two principal maxima appear, separated by a complete minimum and followed by secondary maxima and minima.

By assumption, $\frac{2\pi\alpha' a}{\lambda}$ is small compared to π. It is therefore *a fortiori for points of the object slit that*

$$\xi = \frac{2\pi\alpha' x}{\lambda} \text{ is small compared to } \pi$$

Figure 44



and one obtains the surprising result that the slit itself appears almost completely dark, and that the maxima and minima lie symmetrically on both sides of it.

II. If $\frac{2\pi\alpha' a}{\lambda}$ is large, we need to consider only positive x because the quantity $J_1 - J_2$ changes only its sign for the corresponding negative x, and I thus takes on the same value.

First, suppose

$$\xi = \frac{2\pi\alpha' x}{\lambda} \text{ is small.}$$

Then we can set[lii]

$$J_1 = \frac{\lambda k}{\pi} \int\limits_{\xi}^{\xi + \frac{2\pi\alpha' a}{\lambda}} \frac{\sin w}{w}\, dw = \frac{\lambda k}{\pi} \left\{ \frac{\pi}{2} - \int\limits_0^{\xi} dw\, \frac{w}{w} \right\}$$
$$= \frac{\lambda k}{2} - \frac{\lambda k}{\pi}\xi = \frac{\lambda k}{2} - 2k\alpha' x\,.$$

Likewise,

$$J_2 = \frac{\lambda k}{2} + 2k\alpha' x\,;$$

therefore,

$$J_1 - J_2 = -4k\alpha' x$$

and

$$I = 16k^2\alpha'^2 x^2\,.$$

Therefore, the lowest minimum is found at $x = 0$; on both sides the intensity grows in a steep, parabolic rise. Since we can put

$$\frac{2\pi a\alpha'}{\lambda} + \xi = \infty\,,$$

we can in general write[liii]

$$J_1 - J_2 = -\frac{2\lambda k}{\pi}\int\limits_0^{\xi}\frac{\sin w}{w}\,dw + \frac{\lambda k}{\pi}\int\limits_{-\infty}^{\xi-\frac{2\pi\alpha' a}{\lambda}}\frac{\sin w}{w}\,dw\ . \tag{61}$$

Therefore, if ξ is large compared to π and the observation point is so far from the edge ($x = a$) of the slit that

$$\left|\xi - \frac{2\pi\alpha' a}{\lambda}\right| = \frac{2\pi\alpha'}{\lambda}|x - a|$$

is still large compared to π, then we can set

$$\xi - \frac{2\pi\alpha' a}{\lambda} = -\infty$$

and get

$$J_1 - J_2 = -\frac{2\lambda k}{\pi}\frac{\pi}{2} = -\lambda k\ .$$

We have therefore inside the slit, except in the immediate vicinity of its center and its edges, a nearly *constant brightness*.[liv]

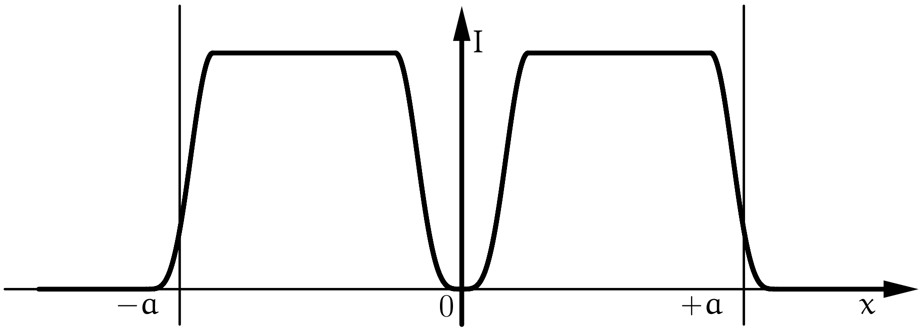
At the edge ($x = a$) we have

$$J_1 - J_2 = -\frac{2\lambda k}{\pi}\frac{\pi}{2} + \frac{\lambda k}{\pi}\frac{\pi}{2} = -\frac{\lambda k}{2}\ .$$

At the edge, therefore, there is only $1/4$ of the intensity that prevails in the slit. If one is *outside* the slit and far enough from its edges, then

$$\xi - \frac{2\pi\alpha' a}{\lambda}$$

Figure 45



is large compared to π and can be set to $+\infty$. We then get

$$J_1 - J_2 = 0 \text{ and therefore also } I = 0\,.$$

The graph of the intensity is therefore largely represented by Fig. 45. This is not entirely correct. In fact, fluctuations of I still appear near the center of the slit $x = 0$ and the edges $x = a$. This can easily be recognized as follows.

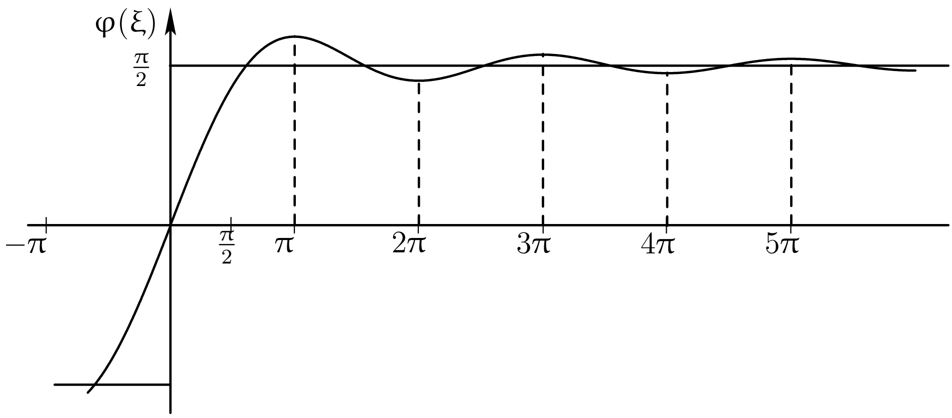
For the sake of simplicity, we base the consideration on the following numerical example:

$$\begin{cases} \dfrac{2\pi\alpha' a}{\lambda} = 50\pi\,, \\ \alpha' = 1 \text{ minute } = \dfrac{1}{60^2}\,, \\ \lambda = 6 \cdot 10^{-4}\,\text{mm}\,, \\ \text{therefore } a = 54\,\text{mm}\,. \end{cases}$$

Since the graph of

$$\varphi(\xi) = \int_0^{\xi} \frac{\sin w}{w}\, dw$$

Figure 46



is the one sketched in Fig. 46, we see that

$$\begin{aligned} \varphi(\xi) \text{ has a maximum for } \quad & \xi = \pi, 3\pi, 5\pi \cdots \\ \text{has a minimum for } \quad & \xi = 2\pi, 4\pi, 6\pi, \text{ etc.} \end{aligned}$$

Now, even for $\xi = \pi$ or $\xi = 2\pi$,

$$\xi - \frac{2\pi\alpha' a}{\lambda} \begin{cases} \text{equal to} & -49\pi \\ \text{or} & -48\pi \end{cases}$$

is still deeply *negative*, so that in $J_1 - J_2$ the second integral is small. The first, on the other hand, is $= \varphi(\xi)$, and therefore we have, according to Eq. 61,

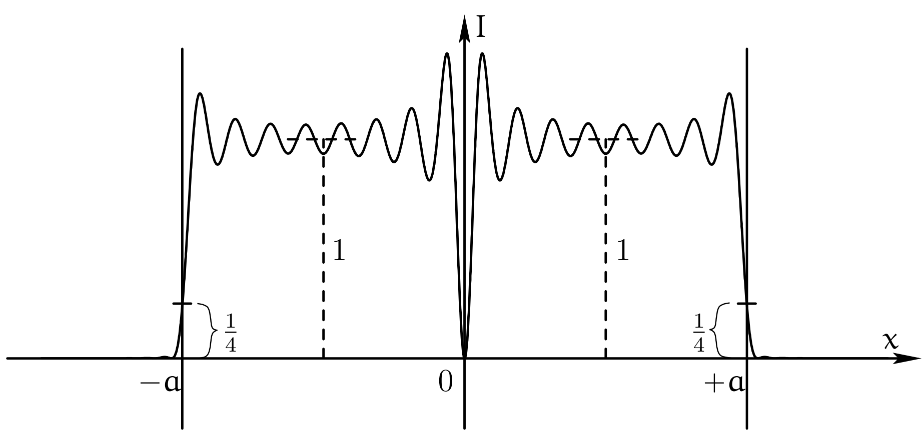
$$J_1 - J_2 = -\frac{2\lambda k}{\pi}\varphi(\xi) \; .$$

I therefore exhibits fluctuations of functional form $[\varphi(\xi)]^2$, so that I assumes a maximum for $\xi = \pi$ and a minimum for $\xi = 2\pi$. The values $\xi = \pi$ and $\xi = 2\pi$, however, correspond to the values

$$x = \frac{\lambda}{2\alpha'} = 1\,\text{mm}$$
$$\text{and} \quad x = \frac{\lambda}{\alpha'} = 2\,\text{mm, respectively.}$$

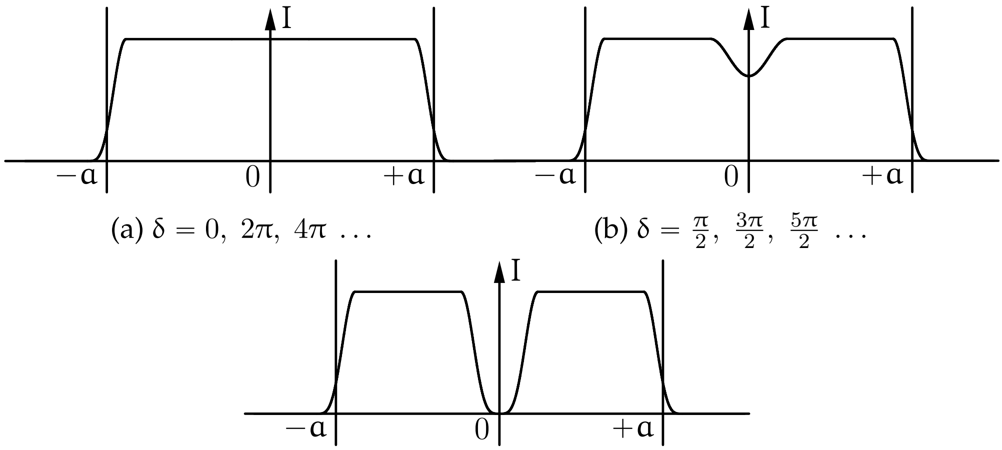
Thus, these "diffraction fringes" close to the center of the slit are still clearly visible. Something quite analogous also occurs at the edges of the slit ($x = \pm a$), as we saw in the previous section. The exact intensity curve will therefore have the form shown in Fig. 47.[lv]

Figure 47



When the phase difference δ of the two halves of the gap increases from 0 to π, the deep minimum in the center only gradually forms (see Figs. 48a, b, and c).

Figure 48



(a) $\delta = 0,\ 2\pi,\ 4\pi\ \ldots$

(b) $\delta = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2}\ \ldots$

(c) $\delta = \pi,\ 3\pi,\ 5\pi\ \ldots$

§24. Slit of finite width with oblique incidence of light

If u is the angle of incidence of the light rays, then the light disturbance at the observation point is

$$\left.\begin{array}{r} s = \dfrac{k}{\lambda} \displaystyle\int_{-b}^{+b} dY\, 2\beta' \frac{\sin 2\pi\beta' \frac{y-Y}{\lambda}}{2\pi\beta' \frac{y-Y}{\lambda}} \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \\ \sin 2\pi \left(\dfrac{t}{T} - \dfrac{X \sin u}{\lambda} \right) \end{array} \right\} . \tag{62}$$

Therefore, for points within the slit zone, we have[lvi]

$$s = k \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \sin 2\pi \left[\frac{t}{T} - \frac{x \sin u}{\lambda} + \frac{(x - X) \sin u}{\lambda} \right]$$

$$= k \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \cos\left(2\pi \sin u \frac{x-X}{\lambda}\right) \sin 2\pi \left(\frac{t}{T} - \frac{x \sin u}{\lambda}\right)$$
$$+ k \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \sin\left(2\pi \sin u \frac{x-X}{\lambda}\right) \cos 2\pi \left(\frac{t}{T} - \frac{x \sin u}{\lambda}\right) .$$

Because the last factors in both integrals do not contain X, we can write

$$\left.\begin{aligned} s &= A \sin 2\pi \frac{t'}{T} + B \cos 2\pi \frac{t'}{T} , \\ \text{where } t' &= t - \frac{x \sin u}{c} \\ A &= k \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \cos\left(2\pi \sin u \frac{x-X}{\lambda}\right) \\ B &= k \int_{-a}^{+a} dX\, 2\alpha' \frac{\sin 2\pi\alpha' \frac{x-X}{\lambda}}{2\pi\alpha' \frac{x-X}{\lambda}} \sin\left(2\pi \sin u \frac{x-X}{\lambda}\right) \end{aligned}\right\} . \tag{63}$$

If we set

$$2\pi\alpha' \frac{x-X}{\lambda} = w ,$$

then we have

$$\left.\begin{aligned} A &= \frac{k\lambda}{\pi} \int_{2\pi\alpha' \frac{x-a}{\lambda}}^{2\pi\alpha' \frac{x+a}{\lambda}} dw \cdot \frac{\sin w}{w} \cdot \cos\left(\frac{\sin u}{\alpha'} w\right) \\ B &= \frac{k\lambda}{\pi} \int_{2\pi\alpha' \frac{x-a}{\lambda}}^{2\pi\alpha' \frac{x+a}{\lambda}} dw \cdot \frac{\sin w}{w} \cdot \sin\left(\frac{\sin u}{\alpha'} w\right) \end{aligned}\right\} \tag{64}$$

and the intensity is $I = A^2 + B^2$. We need to consider only positive values of x since for a switch of x with $-x$, the value of A is unchanged and the sign of B changes, and therefore I remains the same.

To discuss the expression for B, we consider the integral

$$\begin{aligned} B_\infty &= \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dw\,\frac{\sin w}{w}\sin\left(\frac{\sin u}{\alpha'}w\right) \\ &= \frac{1}{\pi}\int\limits_{-\infty}^{0} dw\,\frac{\sin w}{w}\sin\left(\frac{\sin u}{\alpha'}w\right) + \frac{1}{\pi}\int\limits_{0}^{+\infty} dw\,\frac{\sin w}{w}\sin\left(\frac{\sin u}{\alpha'}w\right) . \end{aligned}$$

It can readily be seen that the curve represented by the integrand of the first integral is the symmetrical mirror image of the curve represented by the integrand of the second integral with respect to the w-axis. Therefore, $B_\infty = 0$.[lvii]

To discuss A, we consider the integral

$$A_\infty = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} dw\,\frac{\sin w}{w}\cdot\cos\left(\frac{\sin u}{\alpha'}w\right) . \tag{65}$$

According to earlier developments,[lviii] we have

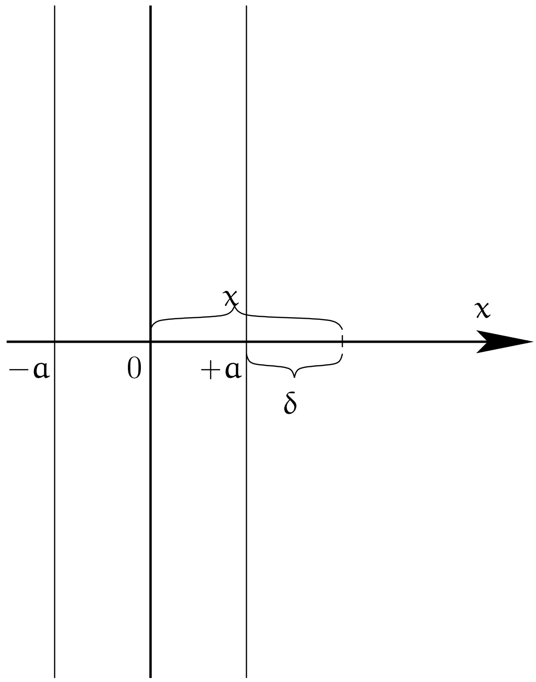
$$\left.\begin{aligned} A_\infty &= 0 \text{ for } \sin u < -\alpha' \text{ and } \sin u > +\alpha' , \\ A_\infty &= 1 \text{ for } \sin u > -\alpha' \text{ and simultaneously } \sin u < +\alpha' \\ A_\infty &= \frac{1}{2} \text{ for } \sin u = \pm\alpha' . \end{aligned}\right\} \tag{66}$$

To compare our integrals A and B with A_∞ and B_∞, we set

$$x = a + \delta ,$$

where δ is the distance of the observation point from the edge of the slit and is to be taken as positive if the observation point varies from

Figure 49



the edge with growing x (Fig. 49). Therefore, δ varies, for positive x, between $-a$ and $+\infty$. Then we have

$$A = \frac{k\lambda}{\pi} \int_{2\pi\alpha' \frac{\delta}{\lambda}}^{2\pi\alpha' \frac{2a+\delta}{\lambda}} dw \, \frac{\sin w}{w} \cdot \cos\left(\frac{\sin u}{\alpha'} w\right)$$

$$B = \frac{k\lambda}{\pi} \int_{2\pi\alpha' \frac{\delta}{\lambda}}^{2\pi\alpha' \frac{2a+\delta}{\lambda}} dw \, \frac{\sin w}{w} \cdot \sin\left(\frac{\sin u}{\alpha'} w\right) \; .$$

For a slit of finite width (a somewhat >1 mm) and a not too small opening angle of the diffracting aperture (α' somewhat $>1°$), $2\pi a\alpha'/\lambda$ is so large compared to π that the upper limit in A and B can be set to ∞. With respect to the lower limit, we again differentiate four cases.

1. The observation point is outside the slit and so far from the edge that one can replace $2\pi\alpha'\delta/\lambda$ by $+\infty$: $A = 0$ and $B = 0$; i.e., the light disturbance is zero regardless of the angle of incidence u. For points far away from the edge, therefore, there is no difference between the phenomena of normal incidence of light and those of oblique incidence of light.

2. The observation point lies within the slit and so far from the edge that $2\pi\alpha'\delta/\lambda$ can be replaced by $-\infty$; then we get

$$B = B_\infty = 0$$
$$A = k\lambda A_\infty \,.$$

   The value of A_∞ still depends on the angle of incidence; in fact, $A = k\lambda$ if $\sin u$ lies between $-\alpha'$ and $+\alpha'$, i.e., if the incident light rays extend through the slit into the diffracting aperture. The total intensity here is then equal to $k^2\lambda^2$. On the other hand, we have $A = 0$ if $\sin u < -\alpha'$ or $\sin u > +\alpha'$, i.e., if the extended light rays no longer hit the diffracing aperture. In this case, the total intensity is therefore equal to zero for all points within the slit but sufficiently far away from the edge.

   If the marginal ray of the incident light beam just hits the edges of the diffracting aperture, then $\sin u = \pm\alpha'$ and $A = \frac{1}{2}k\lambda$; i.e., the total intensity is equal to $k^2\lambda^2/4$.

3. The observation point lies on the edge of the slit. In this case, we have $2\pi\alpha'\delta/\lambda = 0$, and for each incidence angle the values of A and B are half of what they take on in case 2, i.e., when the observation point is located within the slit.[lix]

4. If the observation point lies in the immediate vicinity of the slit edge, we must decompose the integrals A and B in the following way:

$$A = \frac{k\lambda}{\pi}\int_0^{\infty} - \frac{k\lambda}{\pi}\int_0^{2\pi\alpha'\delta/\lambda} = \frac{k\lambda}{2}A_\infty - \frac{k\lambda}{\pi}\int_0^{2\pi\alpha'\delta/\lambda}$$

$$B = \frac{k\lambda}{\pi}\int_0^{\infty} - \frac{k\lambda}{\pi}\int_0^{2\pi\alpha'\delta/\lambda} = -\frac{k\lambda}{\pi}\int_0^{2\pi\alpha'\delta/\lambda} .$$

Of interest is the case where $A_\infty = 0$, i.e., when the extended light rays do *not* hit the diffracting aperture, or when

$$\begin{aligned} &\sin u < -\alpha' \\ \text{or} \quad &\sin u > +\alpha' . \end{aligned}$$

While, as we have seen, in this case the inside of the slit and the slit edges become completely dark, the intensity for points infinitely close to the slit edges retains finite values.

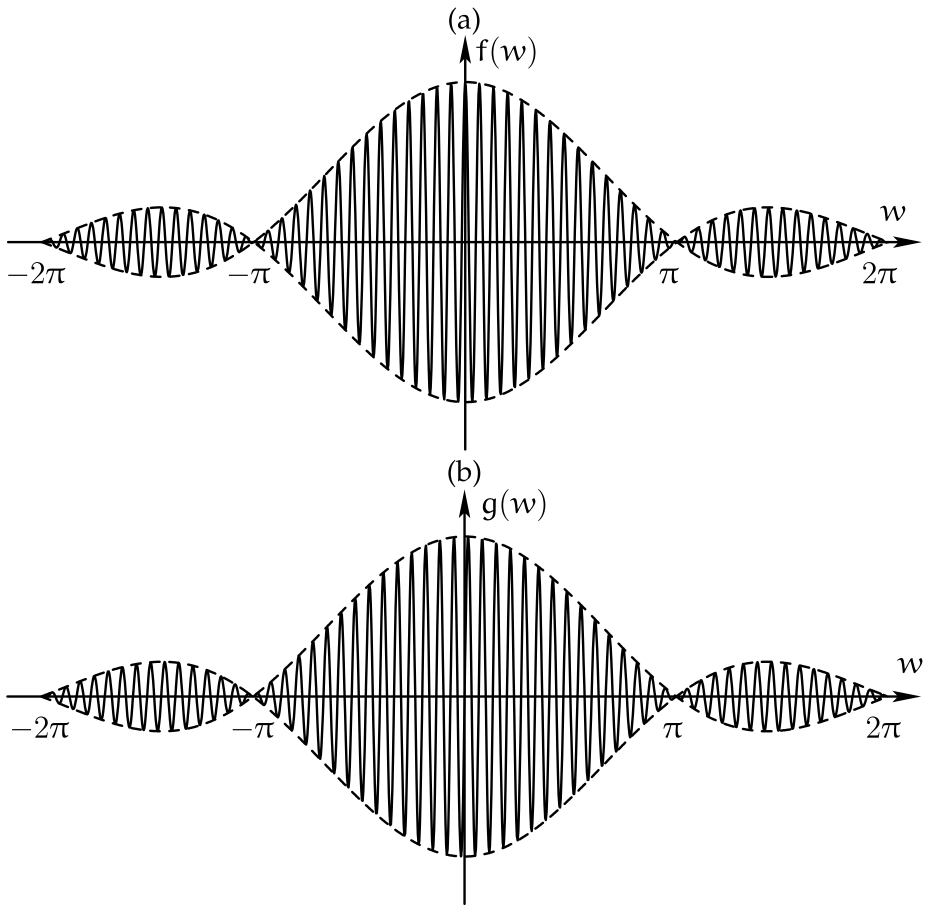
To calculate the intensity distribution close to the edges for various u, we consider the following:

If the value of $\rho = \frac{\sin u}{\alpha'}$ is large, e.g., the magnitude of α' has the value $\sin 1^\circ \simeq \frac{1}{60}$, while u, e.g., $= 30^\circ$, so that $\sin u = \frac{1}{2}$, then the graphs of the functions

$$\begin{aligned} &f(w) = \frac{\sin w}{w}\cos(\rho w) \\ \text{and} \quad &g(w) = \frac{\sin w}{w}\sin(\rho w) \end{aligned}$$

are the ones plotted approximately in Figs. 50a and b.

Figure 50



We observe that the curves $f(w)$ and $g(w)$ intersect the axis $\rho - 1$ times between $w = 0$ and $w = \pi$ at distances $\frac{\pi}{\rho}$. The first intersection after the point $w = 0$ happens for the $f(w)$ curve at $w = \frac{1}{2}\frac{\pi}{\rho}$, and for the $g(w)$ curve at $w = \frac{\pi}{\rho}$. Now the intensity is

$$I = A^2 + B^2 \,,$$

$$\text{where} \qquad A = \text{const} \int_0^{\frac{2\pi\alpha'\delta}{\lambda}} dw \, \frac{\sin w}{w} \cos(\rho w)$$

$$B = \text{const} \int_0^{\frac{2\pi\alpha'\delta}{\lambda}} dw \, \frac{\sin w}{w} \sin(\rho w) \,.$$

If one then moves the boundary $\frac{2\pi\alpha'\delta}{\lambda}$ along the axis of the curves $f(w)$ and $g(w)$ and forms the corresponding areal content represented by A or B, one can easily recognize the following:

With growing $|\delta|$, I executes a series of fluctuations with decreasing amplitude. The minima of the fluctuations lie at locations

$$\frac{2\pi\alpha'\delta}{\lambda} = \pm\frac{2\mathfrak{a}\pi}{\rho} \; (\mathfrak{a} = 0, 1, 2, 3 \ldots) \,,$$

$$\text{i.e.,} \qquad \delta = \pm\frac{\mathfrak{a}\lambda}{\sin u} \,.$$

They maintain a distance $\frac{\lambda}{\sin u}$ from each other. The intensity of the maxima is extremely low.

If, on the other hand, $\sin u$ is only slightly different from α', that is to say $\sin u = \alpha' + \varepsilon$, where ε is small, then, according to simple calculation,[lx]

$$A = \frac{1}{2}\int_0^{\frac{4\pi\alpha'\delta}{\lambda}} \frac{\sin w}{w}\,dw + (\varepsilon)$$

$$B = \int_0^{\frac{2\pi\alpha'\delta}{\lambda}} \frac{\sin^2 w}{w}\,dw + (\varepsilon)\,,$$

where (ε) denotes quantities of the order of ε. Since the curve $\frac{\sin^2 w}{w}$ always runs above the abscissa, B grows everywhere with increasing δ, whereas A simultaneously experiences the known fluctuations. The minima of the intensity thus occur at intervals

$$\delta = \frac{\lambda}{2\alpha'}\,.$$

The maxima of intensity here have finite values (Fig. 51).[lxi]

So far, we have always assumed that the slit is so wide that $\frac{2\pi a\alpha'}{\lambda}$ is large compared to π.

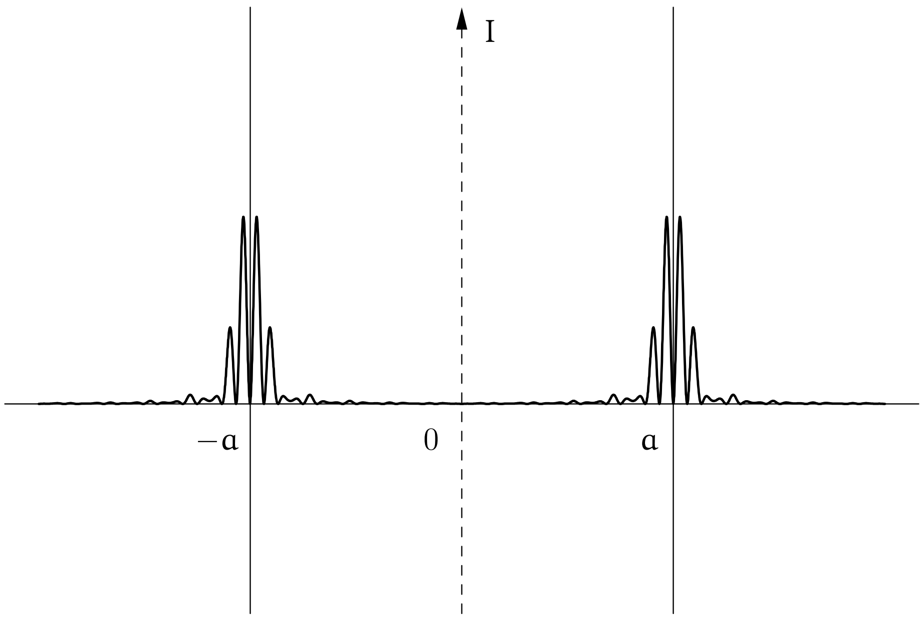
We now proceed to the consideration of a *finite but very narrow* slit by assuming that $\frac{2\pi a\alpha'}{\lambda}$ is small compared to π, thereby gaining a supplement and extension of the already discussed theory of the infinitely narrow slit. In practice, in order to make $\frac{2\pi a\alpha'}{\lambda}$ small compared to π, one must duly reduce α', since, e.g., even for

$$\alpha' = 1^{\circ}$$
$$a = \frac{1}{100}\,\text{mm}$$
$$\lambda = 6\cdot 10^{-4}\,\text{mm}\,,$$

$\frac{2\pi a\alpha'}{\lambda}$ is about $\frac{\pi}{2}$ and still not small compared to π. If we set

$$\begin{cases} \dfrac{2\pi a\alpha'}{\lambda} = \varepsilon \text{ (small)} \\ \dfrac{2\pi\alpha' x}{\lambda} = \xi\,, \end{cases}$$

Figure 51



then the expression for the intensity is

$$I = A^2 + B^2$$

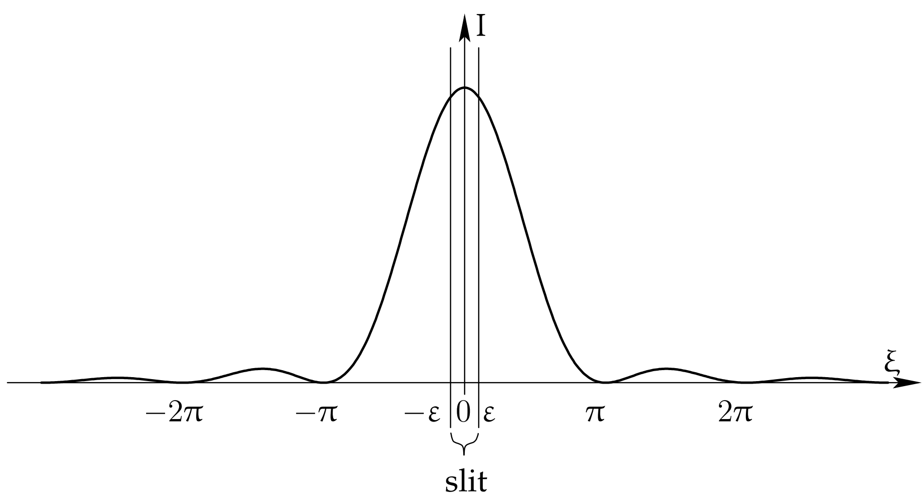
$$\begin{cases} A = \dfrac{k\lambda}{\pi} \displaystyle\int\limits_{\xi-\varepsilon}^{\xi+\varepsilon} dw \, \dfrac{\sin w}{w} \cos\left(\dfrac{\sin u}{\alpha'} w\right) \\ B = \dfrac{k\lambda}{\pi} \displaystyle\int\limits_{\xi-\varepsilon}^{\xi+\varepsilon} dw \, \dfrac{\sin w}{w} \sin\left(\dfrac{\sin u}{\alpha'} w\right) \ . \end{cases}$$

If $\frac{\sin u}{\alpha'}$ is not too large, so that $\varepsilon \frac{\sin u}{\alpha'}$ is small compared to 1, i.e., if we have almost normal incidence, we can expand A and B according to the Taylor series and obtain in a first approximation[lxii]

$$I = 4\varepsilon^2 \frac{k^2\lambda^2}{\pi^2} \left(\frac{\sin\xi}{\xi}\right)^2 .$$

The same value applies to normal incidence, $u = 0$. Thus, in the slit itself ($\xi \simeq 0$) there is an almost constant, strongest brightness; maxima and minima line up symmetrically on both sides of the slit (see Fig. 52).

Figure 52



If the incidence is tilted, i.e., if $\sin u$ has a finite value, then, since α' is very small, the magnitude

$$\rho = \frac{\sin u}{\alpha'}$$

is very large. The graphs of the integrands A and B, i.e., the functions

$$f(w) = \frac{\sin w}{w}\cos(\rho w)$$
$$g(w) = \frac{\sin w}{w}\sin(\rho w)\,,$$

are in this case already represented in Figs. 50a and b.

The graphs of A and B as functions of ξ therefore depend on the ratio of the small interval of integration

$$2\varepsilon = \frac{4\pi a \alpha'}{\lambda}$$

to the likewise small quantity π/ρ, which represents the distance between two successive zero points of the curves $f(w)$ and $g(w)$. We want to distinguish two main cases.

1. Let
$$2\varepsilon = 2\mathfrak{a}\frac{\pi}{\rho}\ (\mathfrak{a} = 1, 2, 3 \ldots)\,.$$

   Then we have

$$2a\sin u = \mathfrak{a}\lambda\ (\mathfrak{a} = 1, 2, 3 \ldots)\,;$$

   i.e., the path difference of the rays striking the edges of the object slit is an integer multiple of the wavelength. It is then for all ξ, as can be easily seen, A and B almost $= 0$, since in the formation of the integrals the adjacent pieces always cancel each other out. *The entire field of vision is therefore dark.* This is natural: the incident light experiences diffraction at the object slit. The principal maximum lies in the extension of the incident rays, that is, below the "diffraction angle" u. The minima lie in the directions

$$\sin u = \frac{\mathfrak{a}\lambda}{2a}\ (\mathfrak{a} = 1, 2, 3 \ldots)\,,$$

and the secondary maxima lie in the directions

$$\sin u = \frac{(2a+1)\lambda}{2 \cdot 2a} \; (a = 0, 1, 2, 3 \ldots) \; .$$

In the considered case, $\sin u = \frac{a\lambda}{2a}$; therefore, a minimum generated by the object slit falls on the diffracting slit (α, β), and the field of view is therefore dark, as deduced above.

2. Let

$$2\varepsilon = (2a+1)\frac{\pi}{\rho} \; (a = 0, 1, 2, 3 \ldots) \; ;$$

then we have

$$2a \sin u = \frac{(2a+1)}{2} \cdot \lambda \; (a = 0, 1, 2, 3 \ldots) \; .$$

In this case, one of the secondary maxima of the diffraction image generated by the object slit falls on the diffracting aperture.

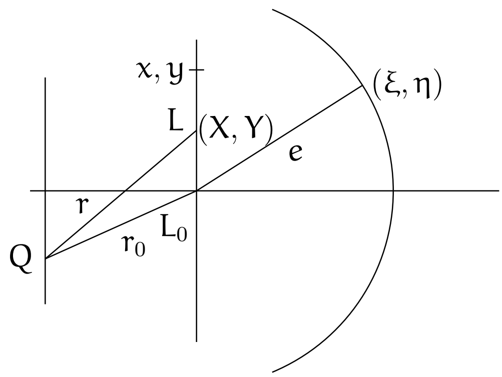
One sees immediately that for $\xi = 0$, that is, in the middle of the slit, A has a value different from zero, which becomes smaller the larger the 2ε, i.e., the more oblique the incidence of light and therefore the higher the order of the maximum that falls on the diffracting aperture. B, on the other hand, is always 0 for $\xi = 0$.

If ξ now grows, A and B periodically assume maxima and minima in rapid succession in such a way that whenever A becomes near 0, B reaches its maximum value and vice versa. At the same time, however, these maximum values decrease from $\xi = 0$ to $\xi = \pi$, and then increase again, thus causing periodic fluctuations in the "wide" intervals of π. Therefore, similar to normal incidence of light, the well-known diffraction pattern will appear, with the principal maximum at the place of the object slit and its secondary maxima and minima symmetrically on both sides, as shown in Fig. 52.

§25. Switching of the order of integration in the calculation of the resulting light disturbance

In what follows, we deal with the general problem: A point source Q (Fig. 53) illuminates the object whose center L_0 lies on the axis of the imaging system. An arbitrary point L of the object has the

Figure 53



coordinates X, Y. The image of the small object is sought using an arbitrary aperture of the imaging system. As before, we introduce as an "intermediate surface" a spherical surface whose points ξ, η have the nearly constant distance e from the individual object points X, Y. Then the light disturbance at a point X, Y of the object on the side facing the intermediate surface can be represented by

$$K\varphi(X,Y)\sin 2\pi\left[\frac{t}{T} - \Psi(X,Y)\right], \tag{67}$$

where $\varphi(X, Y)$ is the transmission coefficient of the object element $dX\,dY$, and $K\varphi(X, Y)$ is the amplitude of the disturbance at the location of the element $dX\,dY$. $\Psi(X, Y)$ can be divided into two parts:

$$\Psi(X,Y) = \frac{r - r_0}{\lambda} + \psi(X,Y)\,.$$

In this case, the factor $\frac{r-r_0}{\lambda}$ takes into account the oblique incidence of the light and $\psi(X, Y)$ the delay of the waves as a result of passing through the object element.

According to earlier results,[lxiii] the sought resulting disturbance at the observation point x, y is then

$$\left.\begin{aligned} S = & \frac{K}{\lambda^2} \iint\limits_{\text{object}} dX\, dY\, \varphi(X, Y) \iint d\xi'\, d\eta' \sin 2\pi \left[\frac{t}{T} - \frac{\xi'(x - X)}{\lambda} \right. \\ & \qquad\qquad\qquad\qquad \left. - \frac{\eta'(y - Y)}{\lambda} - \Psi(X, Y) \right], \\ & \text{where we set } \xi' = \frac{\xi}{e}, \eta' = \frac{\eta}{e}. \end{aligned} \right\} \tag{68}$$

The integration with respect to X, Y extends over the illuminated object, the integration with respect to ξ', η' over the projection of the "effective patch" of the intermediate surface.

In carrying out the integration, one can proceed as before. One integrates first over the intermediate surface (ξ', η') and then over the object (X, Y). The first integration provides, in the object plane,[lxiv] the effect of diffraction of the extent-limiting aperture due to the presence of one object element; the second integration takes into account the extent of the object.

The formation of the image becomes physically clearer if one reverses the order of the integrations and carries out the integration with respect to X, Y first. This immediately provides the effect of diffraction of the illuminated object at the location of the intermediate surface. If the object is, e.g., a grating, then the well-known diffraction spectra occur on the intermediate surface, the positions of which depend on the grating constant and the angle of incidence of the light. After performing the first integration, one can therefore abstract both the light source and the object since both have been replaced by the diffraction spectra appearing on the intermediate surface.

The second integration over ξ', η' has therefore only the role of calculating the interference effect of these diffraction spectra at a point x, y in the object plane.

The resulting phenomenon ("image") is thus the interference effect of a diffraction phenomenon: the primary one being the diffraction phenomenon on the intermediate surface created by the light source and the object, and the secondary one being the effect of interference in the object plane. Only then can one recognize clearly the difference between the image of a self-luminous and an illuminated object.

In the presence of an object of a complicated structure, the evaluation of S is hardly feasible. On the other hand, general rules can be derived that specify under what conditions an "image" similar to the existing object appears, or to which fictitious object instead of the existing one the appearing phenomenon is similar.

To derive these rules, we decompose the expression S into two parts, S_1 and S_2. The first part, S_1, emerges from S if the integration is extended over the entire intermediate surface (hemisphere), i.e., if ξ' and η' take on all values from -1 to $+1$. S_2, however, extends over the entire intermediate surface with the exclusion of the "effective part."

For simplification, we set

$$\left.\begin{array}{c} \dfrac{X}{\lambda} = X'; \ \ \dfrac{Y}{\lambda} = Y'; \ \ \dfrac{x}{\lambda} = x'; \ \ \dfrac{y}{\lambda} = y' \\ \varphi(\lambda X', \lambda Y') = \varphi_1(X', Y'); \qquad \Psi(\lambda X', \lambda Y') = \Psi_1(X', Y') \end{array}\right\} . \qquad (69)$$

We then get

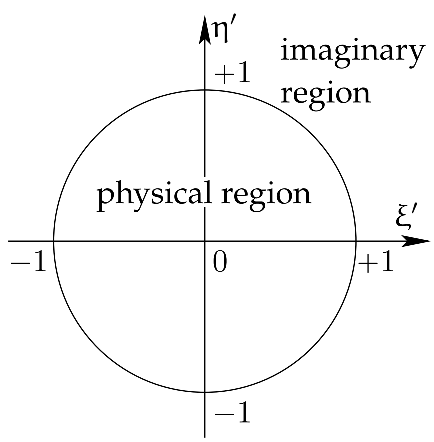
$$\left.\begin{aligned} S_1 &= K \iint\limits_{-1}^{+1} d\xi'\, d\eta' \iint\limits_{\text{object}} dX'\, dY'\, \varphi_1(X', Y') \sin 2\pi \left[\frac{t}{T} - \Psi_1(X', Y') \right. \\ &\qquad\qquad \left. - \xi'(x' - X') - \eta'(y' - Y') \right] \\ S_2 &= S_1 - S\,. \end{aligned}\right\} \tag{70}$$

The variables $\xi' = \frac{\xi}{e}$ and $\eta' = \frac{\eta}{e}$ are sines of the angles, and therefore the following relations are valid:

$$-1 \leqslant \begin{cases} \xi' \\ \eta' \end{cases} \leqslant +1\,. \tag{71}$$

If we represent ξ' and η' as orthogonal coordinates in the $\xi'\eta'$-plane (Fig. 54), then ξ', η' have physical meaning only in the unit

Figure 54



circle around the origin. Outside this circle, the angles to which ξ' and η' belong as sines become imaginary.

Only in the interior of this unit circle does the function contained in S_1,

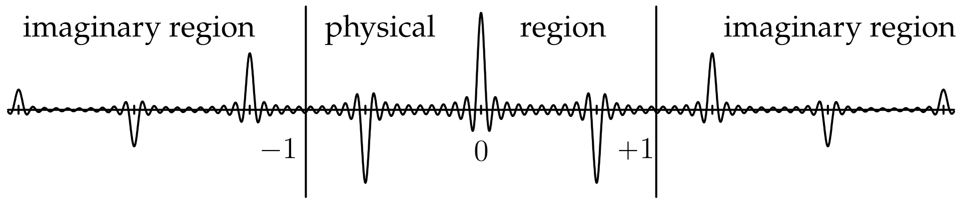
$$f(\xi',\eta') = K \iint\limits_{\text{object}} dX'\, dY'\, \varphi_1(X',Y') \sin 2\pi \left\{ \left[\frac{t}{T} - \Psi_1(X',Y') + \xi' X' + \eta' Y' \right] \right\}, \tag{72}$$

which represents the light disturbance in points ξ, η of the intermediate surface, have *physical significance*. Therefore, we want to call the unit circle in the $\xi'\eta'$-plane the *physical region* and the exterior of the unit circle the *imaginary region*.

Only in the physical region does $f(\xi',\eta')$ have a physical, real meaning. On the other hand, in purely mathematical terms, of course, one can continue the function $f(\xi',\eta')$ into the imaginary region. It is as if one were unaware of the meaning of the variables ξ', η' and treated them as infinitely variable.

For example, if the object is a grating, then part of the function $f(\xi',\eta')$ would be the known grating-generated diffraction image that extends across the hemisphere (intermediate surface) and *breaks off* at its boundaries $\xi' = \pm 1$ and $\eta' = \pm 1$. Mathematically, on the other hand, we can continue the diffraction image with its sharp, gradually extinguishing maxima up to $\xi' = \pm\infty$ and $\eta' = \pm\infty$. The number of maxima that are in the physical region depends on the grating constant and is greater, the larger the grating constant. (See Fig. 55, in which the *amplitudes* of the diffraction maxima are plotted.)[lxv]

Figure 55



If we form the integral

$$S_1^* = K \iint\limits_{-\infty}^{+\infty} d\xi'\, d\eta' \iint\limits_{\text{object}} dX'\, dY'\, \varphi_1(X',Y') \sin 2\pi \left. \begin{array}{l} \left[\dfrac{t}{T} - \Psi_1(X',Y') \right. \\ \qquad \left. -\xi'(x'-X') - \eta'(y'-Y') \right] \end{array} \right\} , \tag{73}$$

which extends over all real *and* imaginary maxima, we shall be able to identify this integral more closely with S_1, the smaller the contribution of $f(\xi',\eta')$ in the imaginary region, and in the example of a grating, the smaller the number of maxima lying in the imaginary region, i.e., the larger the grating constant. Strictly speaking, S_1^* is never equal to S_1. However, if the diffration effect of the object represented by $f(\xi',\eta')$ in the *imaginary region is vanishingly small*, so that almost the entire image of the function $f(\xi',\eta')$ has expanded in the physical region, the equation

$$S_1 = S_1^*$$

represents in praxis a well usable approximation.

We now prove that the expression S_1^* transitions into the expression

$$K\varphi(x,y)\sin 2\pi\left[\frac{t}{T}-\Psi(x,y)\right] ,$$

if the observation point x, y coincides with the object point X, Y, i.e., that S_1^* represents the light disturbance present at the object points x, y on the side of the object that faces the intermediate surface.[lxvi]

For this purpose, we decompose the sine in the integral and write

$$\begin{aligned} S_1^* =&K\iint\limits_{-\infty}^{+\infty} d\xi'\,d\eta' \iint\limits_{\text{object}} dX'\,dY'\,\varphi_1(X',Y')\sin 2\pi\left[\frac{t}{T}-\Psi_1\right] \\ &\qquad\qquad\qquad \cdot\cos 2\pi[\xi'(x'-X')+\eta'(y'-Y')] \\ &-K\iint\limits_{-\infty}^{+\infty} d\xi'\,d\eta' \iint\limits_{\text{object}} dX'\,dY'\,\varphi_1(X',Y')\cos 2\pi\left[\frac{t}{T}-\Psi_1\right] \\ &\qquad\qquad\qquad \cdot\sin 2\pi[\xi'(x'-X')+\eta'(y'-Y')] \,. \end{aligned}$$

If we set $\begin{cases} K\varphi_1(X',Y')\sin 2\pi\left[\frac{t}{T}-\Psi_1\right] = F(X',Y') \\ K\varphi_1(X',Y')\cos 2\pi\left[\frac{t}{T}-\Psi_1\right] = G(X',Y') \,, \end{cases}$

we get

$$\left.\begin{aligned} S_1^* = &\iint\limits_{-\infty}^{+\infty} d\xi'\,d\eta' \iint\limits_{\text{object}} F(X',Y')\,dX'\,dY'\cos 2\pi[\xi'(x'-X')+\eta'(y'-Y')] \\ -&\iint\limits_{-\infty}^{+\infty} d\xi'\,d\eta' \iint\limits_{\text{object}} G(X',Y')\,dX'\,dY'\sin 2\pi[\xi'(x'-X')+\eta'(y'-Y')] \end{aligned}\right\} . \tag{74}$$

We can easily show that

$$\iint\limits_{-\infty}^{+\infty} d\xi'\, d\eta' \iint\limits_{\text{object}} F(X',Y')\, dX'\, dY' \cos 2\pi[\xi'(x'-X') + \eta'(y'-Y')]$$

$$= F(x',y')\text{ , if point } x', y' \text{ lies inside the object,}$$

$$= 0\text{ , if point } x', y' \text{ lies outside the object,}$$

and $$\iint\limits_{-\infty}^{+\infty} d\xi'\, d\eta' \iint\limits_{\text{object}} G(X',Y')\, dX'\, dY' \sin 2\pi[\xi'(x'-X') + \eta'(y'-Y')]$$

$$= 0 \text{ for all locations of point } x', y' \text{ .}$$

This is because the two Fourier theorems apply:[lxvii]

$$\int\limits_{-\infty}^{+\infty} d\xi \int\limits_{A_1}^{A_2} dX\, F(X) \cos 2\pi\xi(x-X) = \begin{cases} F(x)\text{ , if } x \text{ is inside } A_1 \ldots A_2 \\ 0\text{ , if } x \text{ is outside } A_1 \ldots A_2 \end{cases}$$

and

$$\int\limits_{-\infty}^{+\infty} d\xi \int\limits_{A_1}^{A_2} dX\, F(X) \sin 2\pi\xi(x-A) = 0\text{ , for all values of } x.$$

From this, it follows that

$$\int\limits_{-\infty}^{+\infty} d\xi \int\limits_{A_1}^{A_2} dX\, F(X,y) \cos 2\pi\xi(x-X) = F(x,y)\text{ , if } x \text{ is between } A_1 \text{ and } A_2,$$

and

$$\int\limits_{-\infty}^{+\infty} d\eta \int\limits_{B_1}^{B_2} dY\, F(X,Y) \cos 2\pi\eta(y-Y) = F(X,y)\text{ , if } y \text{ is between } B_1 \text{ and } B_2.$$

Therefore, by substitution,

$$\iint_{-\infty}^{+\infty} d\xi\, d\eta \int_{A_1}^{A_2}\int_{B_1}^{B_2} dX\, dY\, F(X,Y)\cos 2\pi\xi(x-X)\cos 2\pi\eta(y-Y)$$

$= F(x,y)$, if x, y lie between $A_1 \ldots A_2$ and $B_1 \ldots B_2$, respectively.

By analogy, we have

$$\iint_{-\infty}^{+\infty} d\xi\, d\eta \int_{A_1}^{A_2}\int_{B_1}^{B_2} dX\, dY\, F(X,Y)\sin 2\pi\xi(x-X)\sin 2\pi\eta(y-Y) = 0$$

for all values of x, y.

By subtracting the last formula from the one before, we get, finally,

$$\left.\begin{aligned} &\iint_{-\infty}^{+\infty} d\xi\, d\eta \int_{A_1}^{A_2}\int_{B_1}^{B_2} dX\, dY\, F(X,Y)\cos 2\pi[\xi(x-X)+\eta(y-Y)] \\ &\qquad = F(x,y) \text{ , when } x \text{ and } y \text{ lie between } A_1 \text{ and } A_2 \\ &\qquad\qquad \text{and between } B_1 \text{ and } B_2\text{, respectively,} \\ &\qquad = 0 \text{ for all other locations of } x, y. \end{aligned}\right\} \tag{75}$$

It can easily be shown in an analogous fashion that we have, additionally,

$$\left.\begin{aligned} &\iint_{-\infty}^{+\infty} d\xi\, d\eta \int_{A_1}^{A_2}\int_{B_1}^{B_2} dX\, dY\, G(X,Y)\sin 2\pi[\xi(x-X)+\eta(y-Y)] \\ &\qquad = 0 \text{ for all locations of } x, y. \end{aligned}\right\} \tag{76}$$

This proves what was already anticipated above that

1.
$$\left.\begin{aligned} S_1^* = F(x', y') &= K\varphi_1(x', y') \sin 2\pi \left[\frac{t}{T} - \Psi_1(x', y')\right] \\ &= K\varphi(x, y) \sin 2\pi \left[\frac{t}{T} - \Psi(x, y)\right] \end{aligned}\right\} \tag{77}$$

if the point x, y lies within the object.

2. $S_1^* = 0$ for all points x, y outside the object.

Therefore, S_1^* *represents the light distribution present on that side of the object* (X, Y) *to be imaged, facing the intermediate surface.*

§26. Pointwise and similar imaging of the object

Referring to the previous paragraph, a pointwise and similar imaging takes place when S can be completely replaced by S_1^*. This is always the case if all the diffraction maxima down to negligible intensity contribute to image formation, i.e., if the aperture of the imaging system (the "effective part" of the intermediate surface) collects all the rays diffracted from the object down to negligible intensity. Thus, there is always an absolute similarity between image and object if the *entire* image of the function $f(\xi', \eta')$ can be expanded within the aperture, but there is dissimilarity if the aperture *does not* collect all diffraction maxima of $f(\xi', \eta')$, i.e., if only parts of the image of the function lie within the aperture.

We shall discuss on which physical quantities the capacity of the system and thus its performance depends. For this, we consider the imaging of a grating. For a given wavelength λ_0 of the incident light, the position of the hth peak is given by the relation

$$\sin u_h = \lambda_0 \frac{h}{n\gamma} \,,$$

where u_h denotes the diffraction angle of the hth maximum, n the index of refraction of the front medium that contains the intermediate surface (immersion fluid), and γ the grating constant.

The number of maxima within the aperture angle U of our system is therefore

$$h = n \sin U \frac{\gamma}{\lambda_0} \,. \tag{78}$$

As we know, the larger the h, the greater the similarity of the image, and we reach ideal similarity for $h = \infty$. For a given grating (γ) and wavelength (λ_0) of the incident light, the number (h) of the image-contributing diffraction maxima that are accepted within the aperture angle is proportional to the product: *index of refraction times sine of the aperture angle.* This product $A = n \cdot \sin U$ has been designated by Abbe as the *numerical aperture* of the system.

Thus, the important theorem follows: *If two systems have the same numerical aperture,*

$$n_1 \sin U_1 = n_2 \sin U_2 \,,$$

they image the same object grating with the same degree of similarity. Only in this way does one actually recognize the meaning of the term numerical aperture introduced by Abbe, that only the product $A = n \cdot \sin U$ determines the similarity of the image, not the aperture angle U of the system. As is well known, for the imaging of self-luminous objects, the numerical aperture is the quantity that alone determines the luminous intensity of the system.

If the aperture angle U of the system for a given λ_0 and γ, as with a dry system ($n = 1$), does not include all the diffraction maxima to vanishing intensity, then the image is a dissimilar one; it can then be transformed into a more similar one if one uses the same system as an immersion system ($n > 1$). As the equation

$$h = n \cdot \sin U \cdot \gamma / \lambda_0$$

shows, the similarity of the image can be increased even more by reducing λ_0.

For a given numerical aperture $A = n \sin U$ of a system with a given wavelength λ_0, the similarity of the grating image is solely due to the grating constant γ. The larger γ becomes, the more diffraction maxima can contribute to image formation, and the greater the similarity. The maximum numerical aperture of a system is reached when $U = 90^o$ and is then

$$A = n \, .$$

Therefore, in this case of *maximum possible performance*,

$$h = n\frac{\gamma}{\lambda_0} \, . \tag{79}$$

If we denote with h_l the last diffraction spectrum of intensity or brightness to be considered in the overall image of the function $f(\xi', \eta')$, the system with $A = n$ will image all gratings with absolute similarity, if

$$\gamma \geqslant \frac{h_l \cdot \lambda_0}{n} \, .$$

§27. Dissimilar imaging of the object

We shall base this investigation on a system with maximum aperture $A = n$, which still images a grating with constant γ with absolute similarity, meaning the satisfaction of the inequality

$$\gamma \geqslant h_l \lambda_0 / n \, ,$$

where h_l is the last diffraction spectrum of intensity still to be considered in the overall image of the function $f(\xi', \eta')$. A grating with a smaller grating constant ($\gamma' < \gamma$) is therefore no longer imaged by the system similarly. If λ_0 has the smallest possible value (photographic waves) and n has the highest possible value (homogeneous immersion), then the grating $\gamma = h_l \lambda_0 / n$ is imaged in an absolutely similar way (a fortiori all gratings with *larger* grating constants), whereas it is physically impossible to image gratings with smaller grating constants ($\gamma' < \gamma$) similarly.

As an example, let us suppose that $\lambda_0 = 350\,\text{nm}$, $n = 1.65$, and $h_l = 10$, assuming that maxima with an intensity less than 1 % of the mean do not contribute to the image. Then the constant of the grating that can still be imaged with absolute similarity ("limit grating") is $\gamma \simeq 2\,\mu\mathrm{m}$.

If we let γ decrease continuously from this limit, more and more maxima of the function $f(\xi', \eta')$ move from the physical region ($\xi', \eta' = -1$ to $+1$) into the imaginary region ($\xi', \eta' < -1$ and $> +1$); i.e., the number of maxima contributing to the image becomes ever smaller and the image becomes more dissimilar. If the grating constant has become so small that only the very center diffraction maximum (principal maximum) lies in the physical region, the dissimilarity reaches its highest degree. We shall denote this maximum dissimilarity as "absolute dissimilarity." It is evidenced by the fact that the image of the structure of the object grating does not show anything, but appears as an almost uniformly luminous area. Only if, in addition to the principal maximum, one of the two adjacent maxima comes into action does the lowest degree of similarity occur; i.e., the image shows interference maxima and minima (structure), and indeed possesses the same number of strokes as the grating.

The lowest degree of similarity is achieved with *central* illumination for

$$\gamma = \frac{\lambda_0}{A}\,, \tag{80}$$

where besides the principal maximum *both* adjacent maxima are contributing. But the same lowest degree of similarity is attained when, apart from the principal maximum, only one of the two adjacent maxima contributes. This can be realized by applying *oblique* illumination, where the grating constant may decrease down to a value of

$$\gamma_m = \frac{\lambda_0}{2A}\,. \tag{81}$$

With this value, the limit of the resolving power of a microscope system is reached.

As is well known, Helmholtz[2] came almost at the same time, albeit in a different way, to the same limit of resolving power.

If one starts by using the full aperture $A = n$ for the grating

$$\gamma < \frac{\lambda_0}{2A}$$

(absolute dissimilarity), with a continuously growing grating constant, new secondary maxima appear continuously and seamlessly in addition to the principal maximum, according to their ordinal number. Here the image always shows just as many interference maxima and minima as the respective grating has "strokes," whereas the intensity decrease from maximum to minimum becomes more and more similar to the intensity distribution in the object grating given by the function $\varphi(X, Y)$. In this way, one finally reaches the "limit grating," which is just about imaged with absolute similarity.

However, with the series of dissimilarities just considered, the variety of dissimilarities is not exhausted. Rather, a large number of variations of dissimilar images of one and the same object grating can be achieved by artificially restricting the aperture or by clipping individual arbitrary and arbitrarily located diffraction maxima. In all these cases, and more generally in the imaging of any microscopic object, a theorem can be derived from our earlier observations, which determines the kind of dissimilarity in each case.

For this, we create a fictitious object (O_f), whose *natural* and *complete* diffraction pattern $[\psi(\xi', \eta')]$ coincides with the diffraction pattern $f(\xi', \eta')$ of the real object (O_r), which was rendered artificially incomplete by stopping down the diaphragm, etc. It is therefore

$$\psi(\xi', \eta')_{\text{complete}} = f(\xi', \eta')_{\text{incomplete}}$$

[2]H. Helmholtz, "The theoretical limit of the resolving power of microscopes," Pogg. Ann, Jubelband 1874,[lxviii] pp. 557–584; Wissenschaftl. Abhandl. Bd. II, pp. 185–212, 1883.

and thus, finally,

$$S[f(\xi', \eta')_{\text{incomplete}}] = S[\psi(\xi', \eta')_{\text{complete}}] = S_1^*[\psi(\xi', \eta')_{\text{complete}}]\,. \quad (82)$$

Thus, the image of the given object O_r in the case of artificial clipping $S[f(\xi', \eta')_{\text{incomplete}}]$ is equal to the absolutely similar image of the fictitious object O_f of the form $S_1^*[\psi(\xi', \eta')_{\text{complete}}]$. For this kind of dissimilarity, we obtain the following general theorem: *The image of the given object* O_r *is identical to the absolutely similar image of that fictitious object* O_f *which would just produce a complete diffraction pattern equal to part of the diffraction pattern of* O_r *accepted by the aperture of the system.*