Extrapolation optimization time power conformable fractional derivative non-linear grey Bernoulli model and its application

Yunsen Zhou*[a], Xinwang Li[b]
[a]School of Transportation and Civil Engineering, Nantong University, Nantong 226019, Jiangsu, China; [b]Zhejiang Jing Gui Technology Co., Ltd., Jiaxin 314399, Zhejiang, China

ABSTRACT

In order to address the structural limitations inherent in the current grey prediction model, this paper proposes the time power conformable fractional derivative non-linear grey Bernoulli model (CFNGBM(p,1) model) based on the traditional grey Bernoulli model, leveraging the theory of fractional calculus. Firstly, by utilizing the computational advantages of conformable fractional derivatives, the grey derivative of the traditional whitening equation is extended from first-order to fractional-order, making the model structure more flexible. Secondly, a time power term is introduced into the model structure to fully capture the non-linear relationship presented in real systems. Additionally, to address the overfitting phenomenon in grey models, an extrapolation optimization mechanism is incorporated into the traditional parameter optimization process by simulating future prediction scenarios. Finally, a case study of monthly production data of photovoltaic glass in the crystalline silicon industry chain demonstrates that the proposed CFNGBM(p,1) model outperforms three other grey Bernoulli models in terms of accuracy, and the use of extrapolation optimization mechanism significantly reduces overfitting.

Keywords: Conformable fractional derivative, time power, extrapolation optimization mechanism, grey system theory

1. INTRODUCTION

The grey prediction model is a core component of grey system theory, with rich achievements[1]. In order to reflect the non-linear characteristics of data sequences, Deng first proposed the grey power prediction model based on the Bernoulli equation (NGBM(1,1) model), which can adapt to different time series by changing the non-linear Bernoulli parameters, thereby exhibiting better fitting performance than traditional grey prediction models[2].

As a foundational model, NGBM(1,1) has room for improvement. In order to further enhance the adaptability of the NGBM(1,1) model to different sequences and achieve a greater capability to adapt to dynamic changes in systems, Wang[3] extended the structural parameters of the NGBM(1,1) model to time-varying parameters. To expand the application scope of the NGBM(1,1) model, Ma et al.[4] proposed a multivariable grey Bernoulli model and validated its feasibility with real-world cases. To further enhance the model's nonlinear fitting capability, Wu et al.[5] introduced the fractional-order accumulation operator into the NGBM(1,1) model. Moreover, Wu et al.[6] combined the NGBM(1,1) model with the NGM(1,1,k,c) model, creating the NGBM(1,1,k,c) model, and validated its feasibility using the annual consumption of fossil energy in five countries. Zheng et al.[7] further developed the NGBM(1,1,k,c) model based on conformable fractional-order accumulation operator and validated its feasibility using China's natural gas production. Wang et al.[8] introduced a novel CFTDNGBM(1,1) model by combining time delay effects, conformable fractional-order accumulation operator, NGBM(1,1) model, arithmetic optimization algorithm, and backward difference operator, successfully applying it to rural economies. Lao et al.[9] capitalized on the advantages of the grey prediction model with a time power term, fractional-order accumulation, and NGBM(1,1) model to develop the DFNGBM(1,1,α) model. Compared to earlier nonlinear grey prediction models, the DFNGBM(1,1,α) model not only meets the requirements of unbiasedness and uniformity but also exhibits a characteristic of prioritizing new information accumulation, thereby demonstrating outstanding performance in short-term forecasting tasks.

Although the optimization methods mentioned above have achieved significant application effects, their whitening differential equations are all first-order. Since first-order derivative models are ideal memory models and are not suitable

*2133110209@stmail.ntu.edu.cn; phone: 19517118190

International Conference on Optics, Electronics, and Communication Engineering (OECE 2024), edited by Yang Yue, Proc. of SPIE Vol. 13395, 133950D · © 2024 SPIE · 0277-786X · Published under a Creative Commons Attribution CC-BY 3.0 License · doi: 10.1117/12.3048719

for describing irregular phenomena, they cannot adjust model parameters based on the actual data characteristics for sequences with large data fluctuations[10]. For this purpose, Wu et al.[11] first extended the whitening differential equation of the grey model from integer-order to fractional-order, establishing the GM(1,1) model incorporating Caputo-type fractional-order derivatives. Subsequently, through analysis and comparison, they confirmed the potential of the GM(1,1) model based on Caputo-type fractional-order derivatives in numerical simulation. Then, subsequent researchers combined Caputo, Grunwald-Letnikov (GL), and Riemann-Liouville (RL) types of fractional-order derivatives with some mainstream grey prediction models, developing more powerful models[12]. Although these fractional-order grey prediction models demonstrate good predictive performance, its computational complexity is high and it is difficult to be widely applied in practical scenarios. Khalil et al.[13] introduced the concept of conformable fractional-order derivatives, which offer a straightforward calculation derivation, making them theoretically easy to handle. Additionally, they can adhere to conventional properties that traditional fractional-order derivatives (such as Caputo, GL, RL, et al.) may not satisfy. Therefore, in recent years, conformable fractional-order derivatives have garnered significant attention. Ma et al.[14] was the first to incorporate the concept of conformable fractional-order derivatives into grey prediction models. Building upon this foundation, Xie et al.[15] constructed a GM(1,1) model based on conformable fractional-order derivatives and revealed the connections between this model and traditional grey prediction models as well as the GM(1,1) model based on Caputo-type fractional-order derivatives. Subsequently, Wu et al.[16] established a multivariate grey prediction model based on conformable fractional-order derivatives and validated its feasibility through examples.

Drawing inspiration from the aforementioned literature, this paper aims to further enhance the adaptability of the grey Bernoulli model to various data sequences and expand the theoretical framework and application scope of grey prediction models. Based on the traditional grey Bernoulli model, we fully exploit the advantages of conformable fractional derivatives by proposing a novel model that integrates conformable fractional derivatives, fractional accumulation, and nonlinear time power terms. And we provide the analytical solution for this model. Furthermore, we propose an extrapolation optimization mechanism to improve existing parameter optimization algorithms for determining the optimal hyperparameters of the model. Finally, we validate the proposed model and apply it to the analysis of the economic development in the photovoltaic silicon industry.

2. SYSTEM DYNAMICS MODELLING

2.1 Conformable fractional derivative

Definition 1: Assumption function $f:\left[t_0,+\infty\right)\to R$, t_0 as the starting point, and $p(0<p\le 1)$ as the order, the conformable fractional derivative is defined as

$$T_{t_0}^{p}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon(t-t_0)^{1-p})-f(t)}{\varepsilon},t>t_0 \tag{1}$$

Lemma 1[17]: If $0<p\le 1$, and f is $p-order$ differentiable function, then

$$T_{t_0}^{p}f(t)=(t-t_0)^{1-p}\frac{df(t)}{dt} \tag{2}$$

2.2 Conformable fractional differential equation

Definition 2[2]: Assuming $X^{(0)}=\left\{x^{(0)}(1),x^{(0)}(2),\cdots,x^{(0)}(n)\right\}$ is a non-negative original sequence, the first-order accumulation generated sequence of $X^{(0)}$ is $X^{(1)}=\left\{x^{(1)}(1),x^{(1)}(2),\cdots,x^{(1)}(n)\right\}$. Then, the sequence generated by the nearest neighbor mean of $X^{(1)}$ is $Z^{(1)}=\{z^{(1)}(1),z^{(1)}(2),\cdots,z^{(1)}(n)\}$, where

$$z^{(1)}(k)=\frac{x^{(1)}(k)+x^{(1)}(k-1)}{2} \tag{3}$$

Definition 3[2]: Assuming $X^{(0)}$, $X^{(1)}$, $Z^{(1)}$ are defined as in Definition 2, then

$$x^{(0)}(k)+az^{(1)}(k)=b(z^{(1)}(k))^{\lambda} \tag{4}$$

which is nonlinear grey Bernoulli model, recorded as NGBM (1,1), its whitening equation is

$$\frac{dx^{(1)}(t)}{dt}+ax^{(1)}(t)=b(x^{(1)}(t))^{\lambda} \tag{5}$$

Obviously, the cumulative order and grey derivative order of traditional grey Bernoulli models are limited to first order, and the model structure does not consider time factors and historical value changes, making it difficult to flexibly simulate sequence features. Based on this, we propose the time power conformable fractional derivative nonlinear grey Bernoulli model.

Definition 4: Assuming $X^{(0)}$ is defined as in definition 2, $X^{(r)}=\left\{x^{(r)}(1),x^{(r)}(2),\cdots,x^{(r)}(n)\right\}$ is an $r-order$ accumulation generated sequence of $X^{(0)}$, where

$$x^{(r)}(k)=\sum_{i=1}^{k}\frac{\Gamma(r+k-i)}{\Gamma(k-i+1)\Gamma(r)}x^{(0)}(i),k=1,2,\cdots,m \tag{6}$$

Definition 5: Assuming $X^{(0)}$, $X^{(r)}$ are defined as in definition 2, then

$$T_0^p x^{(r)}(t)+ax^{(r)}(t)=(b\cdot t^{\alpha}+c)[x^{(r)}(t)]^{\eta},p\in\left(0,1\right],t\in\left(0,T\right] \tag{7}$$

is whitening differential equation of the time power conformable fractional derivative non-linear grey Bernoulli model. Where a represents the development coefficient, $b\cdot t^{\alpha}+c$ is the time power term, b denotes the coefficient of the time power and c stands for the adjustment parameter.

According to Lemma 1, equation (7) can be expressed as:

$$\frac{dx^{(r)}(t)}{dt}+at^{p-1}x^{(r)}(t)=t^{p-1}(b\cdot t^{\alpha}+c)[x^{(r)}(t)]^{\eta} \tag{8}$$

Multiply both sides of the above equation simultaneously $[x^{(r)}(t)]^{-\eta}$, we can obtain

$$\frac{dx^{(r)}(t)}{dt}[x^{(r)}(t)]^{-\eta}+at^{p-1}x^{(r)}(t)[x^{(r)}(t)]^{-\eta}=t^{p-1}(b\cdot t^{\alpha}+c)$$

By simplifying the above formula with $Q^{(r)}(t)=[x^{(r)}(t)]^{1-\eta}$, we can obtain

$$\frac{dQ^{(r)}(t)}{dt}+a(1-\eta)\cdot t^{p-1}Q^{(r)}(t)=b(1-\eta)\cdot t^{\alpha+p-1}+c(1-\eta)\cdot t^{p-1} \tag{9}$$

By approximating derivatives through backward differentiation

$$\left.\frac{dQ^{(r)}(t)}{dt}\right|_{t=k}\approx\frac{Q^{(r)}(k)-Q^{(r)}(k-\Delta t)}{\Delta t}$$

And $\beta_1=a(1-\eta),\beta_2=b(1-\eta),\beta_3=c(1-\eta)$, $G^{(r)}(k)$ is set as the trapezoidal approximation of sequence $Q^{(r)}(k)$. We can obtain the discrete form of equation (9)

$$Q^{(r-1)}(k)+\beta_1\cdot k^{p-1}G^{(r)}(k)=\beta_2\cdot k^{\alpha+p-1}+\beta_3\cdot k^{p-1} \tag{10}$$

We refer to equation (10) as the time power conformable fractional derivative non-linear grey Bernoulli model, denoted as CFNGBM(p,1).

Assuming $X^{(0)}$, $X^{(r)}$ are defined as in definition 2, m is the number of modeled samples, the parameter estimation of the CFNGBM(p,1) model can be obtained as follows:

$$\hat{u}=(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3)^{\mathrm{T}}=(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y \tag{11}$$

where

$$B=\begin{pmatrix} -2^{p-1}G^{(r)}(2) & 2^{\alpha+p-1} & 2^{p-1} \\ -3^{p-1}G^{(r)}(3) & 3^{\alpha+p-1} & 3^{p-1} \\ \vdots & \vdots & \vdots \\ -m^{p-1}G^{(r)}(m) & m^{\alpha+p-1} & m^{p-1} \end{pmatrix}, Y=\begin{pmatrix} Q^{(r)}(2) \\ Q^{(r)}(3) \\ \vdots \\ Q^{(r)}(m) \end{pmatrix}$$

2.3 Solution of the CFNGBM(p,1) model

If $p\in(0,1]$, and $\hat{u}=(\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3)^{\mathrm{T}}$ is identification values of CFNGBM(p,1) model parameter columns and under the condition that the initial value condition $\hat{x}^{(r)}(1)=x^{(0)}(1)$ is met, then when $k=2,3,\cdots,m,\cdots$, the discrete iteration time response equation and final restoration expression of CFNGBM(p,1) are as follows.

(I) The discrete iteration time response equation of CFNGBM(p,1) model is

$$\hat{Q}^{(r)}(k)=h_1(k)\cdot\hat{Q}^{(r)}(k-1)+h_2(k)+h_3(k) \tag{12}$$

where

$$h_1(k)=\frac{(0.5\beta_1\cdot k^{p-1}-1)}{(1+0.5\beta_1\cdot k^{p-1})}, h_2(k)=\frac{\beta_2\cdot k^{\alpha+p-1}}{(1+0.5\beta_1\cdot k^{p-1})}, h_3(k)=\frac{\beta_3\cdot k^{p-1}}{(1+0.5\beta_1\cdot k^{p-1})}$$

(II) The final reduction expression of the CFNGBM(p,1) model is

$$\hat{x}^{(0)}(k)=\sum_{i=0}^{k-1}(-1)^i\frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)}\left[\hat{Q}^{(r)}(k-i)\right]^{\eta-1}, k=1,2,\cdots,m,\cdots \tag{13}$$

3. PARAMETER OPTIMIZATION AND MODEL EVALUATION

3.1 Extrapolation optimization mechanism

From the model's time response, it is evident that the undetermined parameters directly influence the predictive accuracy of CFNGBM (p,1). Therefore, it is necessary to establish an optimization model to optimize the parameters. However, traditional optimization strategies based on a single training set often lead to overfitting, significantly affecting the model's generalization ability. To address this issue, this paper proposes an extrapolation optimization mechanism, dividing the dataset into three parts: (1) the training dataset, (2) the extrapolation optimization dataset, and (3) the testing dataset. By utilizing the union of the training set and extrapolation set for parameter optimization, the optimization process not only simulates the fitting process but also simulates the prediction scenario. This optimization mechanism helps reduce overfitting and greatly enhances model effectiveness. The specific improvement mechanism is shown in Figure 1.

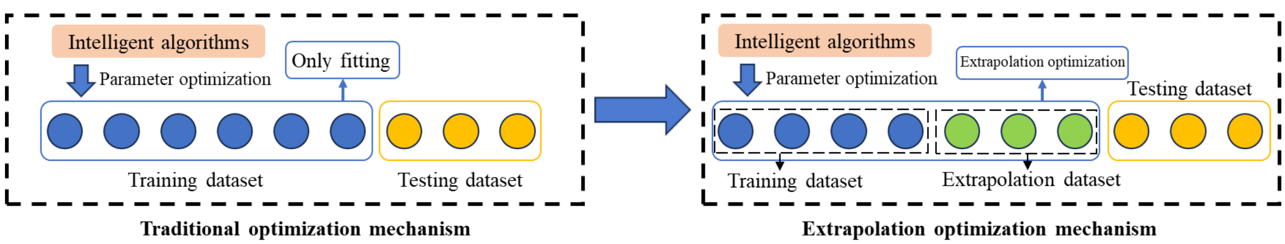

Figure 1. Principle of improved extrapolation parameter optimization mechanism.

3.2 Evaluation indicators

To evaluate the predictive performance of the grey model, we chosen evaluation metrics include the absolute percentage error (APE) and mean absolute percentage error (MAPE). The specific formula is as follows:

$$APE(k)=\frac{\left|\hat{x}^{(0)}(k)-x^{(0)}(k)\right|}{x^{(0)}(k)}\times 100\%$$

$$TMAPE = \frac{1}{m}\sum_{i=1}^{m} APE(k)$$
$$EMAPE = \frac{1}{l-m}\sum_{i=m+1}^{l} APE(k)$$
$$PMAPE = \frac{1}{n-l}\sum_{i=l+1}^{n} APE(k)$$

where, m denote the training set size, $l-m$ denote the extrapolation size, and $n-l$ denote the testing set size. Then we set the training MAPE error as TMAPE, extrapolation optimization MAPE error as EMAPE, and predictive MAPE error as PMAPE separately.

Further a nonlinear programming model is established as follows:

$$\arg\min_{p,r,\alpha,\eta} \frac{1}{l}\sum_{i=1}^{l} APE(k),$$
$$\begin{cases} \hat{u} = (\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3)^{\mathrm{T}} = (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}Y, \\ Q^{(r)}(k) = h_1(k)\cdot Q^{(r)}(k-1) + h_2(k) + h_3(k), \\ \hat{x}^{(0)}(k) = \sum_{i=0}^{k-1}(-1)^i \frac{\Gamma(r+1)}{\Gamma(i+1)\Gamma(r-i+1)}\left[\hat{Q}^{(r)}(k-i)\right]^{\eta-1}, \\ p \in (0,1], k = 2,\cdots,m,\cdots n. \end{cases} \tag{14}$$

Due to the nonlinearity in equation (13), conventional methods for solving it are challenging. In this paper, the particle swarm optimization algorithm is employed to rapidly obtain the model parameters.

4. CASE ANALYSIS OF CRYSTAL SILICON INDUSTRY

4.1 Research significance and data collection

As a vital component of the crystalline silicon industry, photovoltaic glass production directly impacts the production progress of photovoltaic components and the operation of the supply chain. Through time-series forecasting analysis of photovoltaic glass production, enterprises can further optimize production planning and scheduling arrangements. This ensures a balance between production capacity and market demand, mitigating the risk of overcapacity or shortages. Moreover, enterprises can strategically manage inventory levels to prevent excessive or insufficient stockpiling, thereby enhancing production efficiency and resource utilization.

To validate the effectiveness and practicality of the CFNGBM(p,1) model, this section establishes the traditional NGBM(1,1) model[2], DFNGBM(1,1,α) model[9], CFGBM(μ,1) model[18], and CFNGBM(p,1) model. The case study of monthly production of photovoltaic glass was conducted, with data sourced from the Shanghai Nonferrous Metals Information Network (https://data-pro.smm.cn/). This section conducts a case analysis using monthly photovoltaic glass production data from January 2022 to March 2024.

4.2 Photovoltaic glass production

Two experiments are conducted in total: the conventional experiment involving fitting and prediction, and the extrapolated optimization experiment. The raw data is recorded in https://github.com/Zhou-YunSen/Photovoltaic-glass. For the conventional experiment, the original dataset had a sample size of 27, with 24 samples used for the testing set and 3 samples for the validation set. In the hyperparameter optimization experiment, the fitting dataset comprised 21 samples, the extrapolation testing set comprised 3 samples, and the prediction testing set comprised 1 sample. In order to clearly demonstrate the experimental errors of each model, Figure 2 presents the simulation error and prediction error of all models under conventional experiment. Table 1 displays the conventional experiment results of each model on monthly photovoltaic glass production data.

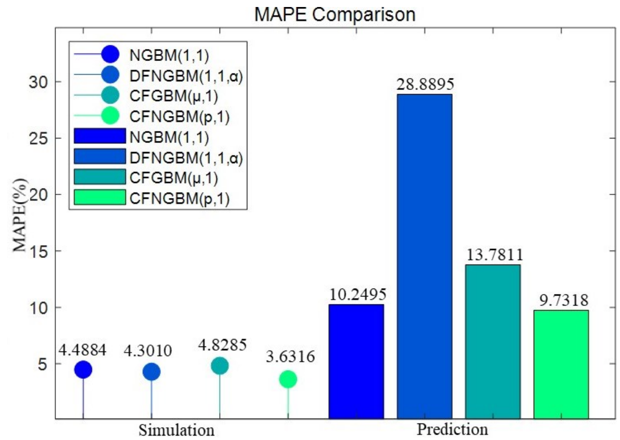

Figure 2. The simulation error and prediction error of all models under conventional experiment.

Table 1. Conventional experiment results of each model on monthly photovoltaic glass production data.

MAPE	Sample size	NGBM(1,1)[2]	DFNGBM(1,1,α)[9]	CFGBM(μ,1)[18]	CFNGBM(p,1)
Simulation MAPE	24	4.4884	4.3010	4.8285	3.6316
Prediction MAPE	3	10.2495	28.8895	13.7811	9.7308

In this experiment, particle swarm optimization (PSO) algorithm is utilized for parameter tuning. The number of particles is set to 500, with inertia weight, cognitive random coefficient, and social random coefficient set at 0.2, 0.6 and 0.6, respectively. The maximum number of iterations is set to 300, and the number of parameters to be optimized can vary depending on the model requirements. In routine experiments of case studies, the optimal value of the λ parameter for the NGBM model is 0.0836. For the DFNGBM(1,1,α) model, the optimal values of parameters r, λ and α are 1.0821, 1.7992 and -2.2369, respectively. Meanwhile, for the CFGBM(μ,1) model, the optimal values of parameters λ and μ are 0.8541 and 0.2214, respectively. Lastly, for the proposed CFNGBM(p,1) model, the optimal values of parameters p, r, η, and α are 0.3412, 0.7400, 1.8812 and 0.2077 respectively.

Analysis of the data in Table 1 reveals a significant disparity between the model's predictive accuracy and its fitting precision. Clearly, when solely prioritizing fitting precision, prevalent discrepancies between predictive and fitting errors are often encountered and the risk of overfitting is notably high. Therefore, it is imperative to consider improvements to the current optimization mechanism. Furthermore, the enhanced stepwise optimization mechanism is embedded into the existing optimization fitting algorithm for experimentation. The specific MAPE results for different stages of each model are presented in Table 2. In Extrapolation experiments of case studies, the optimal value of the λ parameter for the NGBM model is 0.2736. For the DFNGBM(1,1,α) model, the optimal values of parameters r, λ and α are 0.5779, 1.2169 and 0.0133, respectively. Meanwhile, for the CFGBM(μ,1) model, the optimal values of parameters λ and μ are 0.8990 and 0.3499, respectively. Lastly, for the proposed CFNGBM(p,1) model, the optimal values of parameters p, r, η and α are 0.8910, -0.6024, -1.1603 and 3.4733, respectively. In order to clearly demonstrate the experimental errors of each model, Figure 3 presents the comparison between conventional simulation errors and extrapolated simulation errors. Figure 4 presents comparison of conventional prediction error and extrapolation prediction error.

The analysis of the experimental results from Table 2 reveals that the incorporation of a hyper-step optimization mechanism into the prediction algorithm significantly reduces the phenomenon of model overfitting. Moreover, this improved mechanism can be applied not only to the model proposed in this paper but also to other traditional grey forecasting models, demonstrating high practicality. Furthermore, the CFNGBM(p,1) model proposed in this paper exhibits the highest experimental accuracy both in fitting and extrapolation stages, enabling precise prediction of monthly production of photovoltaic glass.

Table 2. Extrapolated experiment results of each model on monthly photovoltaic glass production data.

Time	Raw data	NGBM(1,1)[2]	DFNGBM(1,1,α)[9]	CFGBM(μ,1)[18]	CFNGBM(p,1)
2022-01	100.4	100.4000	100.4000	100.4000	100.4000
2022-02	94.9	107.1894	93.8073	89.3734	99.1598
2022-03	103.8	115.1089	107.4203	102.0738	104.6374
2022-04	115.4	121.7245	117.8913	112.8659	112.6936
2022-05	137.2	127.7555	125.9698	122.4032	121.1098
2022-06	133.5	133.4847	132.9213	131.0504	129.3819
2022-07	135.6	139.0546	139.3756	139.0296	137.3829
2022-08	141.8	144.5482	145.5889	146.4870	145.0849
2022-09	161.2	150.0184	151.6627	153.5243	152.4906
2022-10	151.9	155.5020	157.6414	160.2156	159.6112
2022-11	162.5	161.0255	163.5484	166.6165	166.4595
2022-12	167.8	166.6094	169.3991	172.7701	173.0467
2023-01	156.7	172.2701	175.2050	178.7106	179.3818
2023-02	169.7	178.0208	180.9754	184.4653	185.4710
2023-03	195.3	183.8731	186.7184	190.0569	191.3177
2023-04	204.2	189.8370	192.4405	195.5040	196.9230
2023-05	205.1	195.9214	198.1476	200.8225	202.2851
2023-06	207.4	202.1346	203.8447	206.0257	207.4000
2023-07	216.5	208.4840	209.5362	211.1251	212.2610
2023-08	223.3	214.9771	215.2262	216.1307	216.8591
TMAPE (%)		5.9591	3.6612	4.0692	**3.5778**
2023-10	227.02	228.4210	226.6148	225.8940	225.2162
2023-11	220.56	235.3851	232.3193	230.6660	228.9427
2023-12	235.07	242.5193	238.0342	235.3732	232.3405
EMAPE (%)		5.1021	4.9652	4.8680	**4.6624**
Simulation MAPE		4.3461	3.3117	3.6336	3.4779
2024-01	234.76	249.8297	243.7617	240.0210	235.3838
2024-02	222.7	257.3228	249.5039	244.6137	238.0414
2024-03	243.61	265.0049	255.2628	249.1561	240.2757
PMAPE (%)		3.7546	3.8904	3.3306	**2.0762**

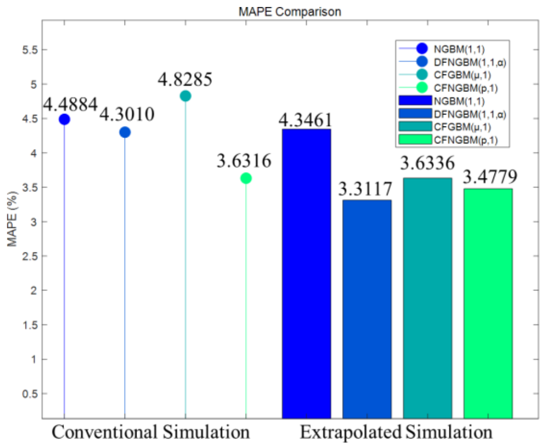

Figure 3. The conventional simulation error and extrapolated simulation error of all models.

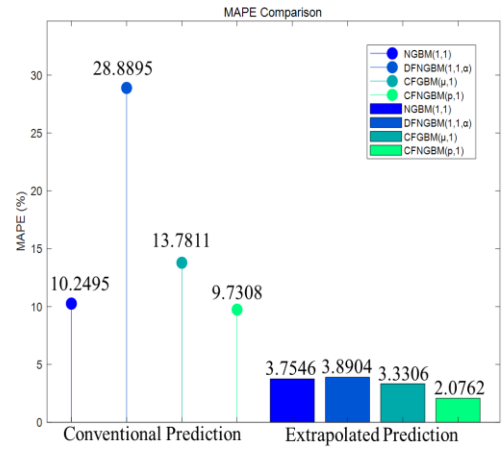

Figure 4. The conventional prediction error and extrapolated prediction error of all models.

5. CONCLUSION

This paper presents a conformable fractional-order derivative of the time-varying grey Bernoulli model, deriving analytical expressions for the discrete iteration of the time response function. It extends the application of grey theory to fractional-order systems and nonlinear science. Furthermore, an extrapolation optimization mechanism is proposed and embedded into the construction of various grey prediction models. In contrast to conventional parameter optimization mechanisms, the proposed extrapolation optimization can effectively mitigate overfitting caused by a simple training set in a straightforward manner. Ultimately, the CFNGBM(p,1) model is applied to the photovoltaic glass monthly production data in the crystalline silicon industry chain, yielding excellent results. It is noteworthy that the extrapolation optimization training mechanism proposed in this paper can not only be applied to grey models and their theories but also has potential applications in areas such as black-box models like neural networks in the future.

REFERENCES

[1] Deng, J. L., [Fundamentals of Grey Theory], Huazhong University of Science and Technology Press, Wuhan, 300-309 (2002).

[2] Chen, C. I., "Application of the novel nonlinear grey Bernoulli model for forecasting unemployment rate," Chaos, Solitons & Fractals 37(1), 278-287 (2008).

[3] Wang, Z. X., "GM(1,1) power model with time-varying parameters and its application," Control and Decision 29(10),1828-1832 (2014).

[4] Ma, X., Liu, Z. B. and Wang, Y., "Application of a novel nonlinear multivariate grey Bernoulli model to predict the tourist income of China," Journal of Computational and Applied Mathematics 347, 84-94 (2019).

[5] Wu, W. Q., Ma, X., Zeng, B., et al., "Forecasting short term renewable energy consumption of China using a novel fractional nonlinear grey Bernoulli model," Renewable Energy 140, 70-87 (2019).

[6] Wu, W. Q., Ma, X., Zeng, B., et al., "A novel grey Bernoulli model for short-term natural gas consumption forecasting," Applied Mathematical Modelling 84, 393-404 (2020).

[7] Zheng, C. L., Wu, W. Z., Xie, W. L., et al., "A MFO based conformable fractional nonhomogeneous grey Bernoulli model for natural gas production and consumption forecasting," Applied Soft Computing 99, 106891 (2021).

[8] Wang, R., Liu, H., Yang, Q. W., et al., "A new conformable fractional-order time-delay grey Bernoulli model with the arithmetic optimization algorithm and its application in rural regional economy," Journal of Mathematics 2023, 2017167 (2023).

[9] Lao, T. F. and Sun, Y. R., "Predicting the production and consumption of natural gas in China by using a new grey forecasting method," Mathematics and Computers in Simulation 202, 295-315 (2022).

[10] Kang, Y. X., Mao, S. H., Zhang, Y. H., et al., "Fractional derivative multivariable grey model for nonstationary sequence and its application," Journal of Systems Engineering and Electronics 31(5), 1009-1018 (2020).
[11] Wu, L. F., Liu, S. F. and Yao, L. G., "Grey model with Caputo fractional order derivative," Systems Engineering-Theory & Practice 35(5),1311-1316 (2015).
[12] Wu, W. Q., Ma, X., Wang, Y., et al., "Research on a novel fractional GM(α,n) model and its applications," Grey Systems (Theory and Application) 9(3), 356-373 (2019).
[13] Khalil, R., Horani, M., Yousef, A., et al., "A new definition of fractional derivative," Journal of Computational and Applied Mathematics 264, 65-70 (2014).
[14] Ma, X., Wu, W. Q., Zeng, B., et al., "The conformable fractional grey system model," ISA Transactions 96, 255-271 (2020).
[15] Xie, W. L., Liu, C. X., Wu, W. Z., et al., "Continuous grey model with conformable fractional derivative," Chaos, Solitons & Fractals 139, 110285 (2020).
[16] Wu, W. Q., Ma, X., Zeng, B., et al., "A novel multivariate grey system model with conformable fractional derivative and its applications," Computers & Industrial Engineering 164, 107888 (2022).
[17] Wang, S., Jiang, W., Sheng, J. L., et al., "Ulam's stability for some linear conformable fractional differential equations" Advances in Difference Equations 2020(1), 1-18 (2020).
[18] Luo, S. and Zeng, L., "A grey Bernoulli model based on conformable fractional derivatives," Journal of Zhejiang University (Science Edition) 51(02), 196-204 (2024).