Infinite Series and Improper Integrals

Infinite series are important in almost all areas of mathematics and engineering. In addition to numerous other uses, they are used to define certain functions and to calculate accurate numerical estimates of the values of these functions. Most of the infinite series that we encounter in practice are known as **power series**.

In calculus the primary problem is deciding whether a given series converges or diverges. In practice, however, the more crucial problem may actually be summing the series. If a convergent series converges too slowly, the series may be worthless for computational purposes. On the other hand, the first few terms of a divergent asymptotic series in some cases may give excellent results.

Improper integrals are used in much the same fashion as infinite series, and, in fact, their basic theory closely parallels that of infinite series. They are often classified as belonging to **improper integrals of the first kind** or **improper integrals of the second kind**.

Series of Constants

If to each positive integer n we can associate a number S_n, then the ordered arrangement $S_1, S_2, \ldots, S_n, \ldots$ is called an **infinite sequence**. If $\lim_{n\to\infty} S_n = S$ (finite), the infinite sequence is said to **converge** to S; otherwise it **diverges**.

Infinite series: $u_1 + u_2 + \cdots + u_k + \cdots = \sum_{k=1}^{\infty} u_k$.

If $S_n \to S$ as $n \to \infty$, where S_n is the sequence defined by $S_n = u_1 + u_2 + \cdots + u_n$, then the series converges to S; otherwise it diverges. The letter "k" used in the summation is called an **index**; any other letter like "n" can also be used.

Geometric series: The special series

$$1 + r + r^2 + \cdots + r^k + \cdots = \sum_{k=0}^{\infty} r^k$$

is an example series where r is the common ratio. It has been shown that

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}, \quad r \neq 1 \text{ and } \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1 .$$

Harmonic series: Although it diverges, the particular series $\sum_{n=1}^{\infty} \frac{1}{n}$ is quite important in theoretical work.

Alternating series: Series of the form

$$\sum_{n=0}^{\infty} (-1)^n u_n, \quad u_n > 0, \quad n = 1, 2, 3, \ldots .$$

Alternating harmonic series: The special alternating series defined by $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ differs from the harmonic series above in that this series converges.

Operations with Series

- The series Σu_n is said to **converge absolutely** if the related series $\Sigma |u_n|$ converges.
- If Σu_n converges but the related series $\Sigma |u_n|$ diverges, then Σu_n is said to **converge conditionally**.

In some applications the need arises to combine series by operations like **addition** and subtraction, and **multiplication**. The following rules of algebra apply:

1. The sum of an absolutely convergent series is independent of the order in which the terms are summed.
2. Two absolutely convergent series may be added termwise and the resulting series will converge absolutely.
3. Two absolutely convergent series may by multiplied and the resulting series will also converge absolutely.

Forming the product of two series leads to **double infinite series** of the form

$$\sum_{m=0}^{\infty} a_m \sum_{k=0}^{\infty} b_k = \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A_{m,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,k} \, .$$

Note that the result on the right is a single infinite series of finite sums rather than a product of two infinite series. Another product of series leads to the identity

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} A_{m,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A_{n-k,k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A_{n-2k,k} \quad ,$$

where

$$\left[\frac{n}{2}\right] = \begin{cases} n/2, & n \text{ even} \\ (n-1)/2, & n \text{ odd} \end{cases} .$$

Factorials and Binomial Coefficients

A product of consecutive positive integers from 1 to n is called **factorial *n*** and denoted by

$$n! = 1\cdot 2\cdot 3\cdots n, \quad n = 1,2,3,\dots \ .$$

The fundamental properties of the factorial are

$$\begin{aligned} &0! = 1 \\ &n! = n(n-1)!, \quad n = 1,2,3,\dots \end{aligned} .$$

The **binomial formula**, or **finite binomial series**, is defined by

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \ .$$

The special symbol

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n = 0,1,2,\dots; \quad k = 0,1,2,\dots,n \ ,$$

is called the **binomial coefficient**. Basic identities for the binomial coefficient include the following:

$$(a) \quad \binom{n}{0} = \binom{n}{n} = 1$$

$$(b) \quad \binom{n}{1} = \binom{n}{n-1} = n$$

$$(c) \quad \binom{n}{k} = \binom{n}{n-k}$$

$$(d) \quad \binom{r}{0} = 1, \quad \binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}, \ k = 1,2,3,\dots$$

$$(e) \quad \binom{-r}{k} = (-1)^k \binom{r+k-1}{k}$$

Factorials and Binomial Coefficients Example

Use the definition of the binomial coefficient and properties of factorials to show that

$$\binom{-1/2}{n} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}, \quad n = 0, 1, 2, \ldots .$$

Solution:

From definition, we obtain

$$\begin{aligned}\binom{-1/2}{n} &= \frac{(-1/2)(-3/2)\cdots(1/2-n)}{n!} \\ &= \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}\end{aligned}.$$

Multiplying the numerator and denomination on the right by the product $2\cdot 4\cdot 6\ldots(2n)$ and rearranging the numerator terms, we find

$$\begin{aligned}\binom{-1/2}{n} &= \frac{(-1)^n 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdots(2n-1)\cdot(2n)}{2^n 2\cdot 4\cdot 6\cdots(2n)n!} \\ &= \frac{(-1)^n (2n)!}{2^n 2^n \cdot 1\cdot 2\cdot 3\cdots n\cdot n!}\end{aligned},$$

which can be written as

$$\binom{-1/2}{n} = \frac{(-1)^n (2n)!}{2^{2n}(n!)^2}, \quad n = 0, 1, 2, \ldots .$$